Quantum Field Theory and Cosmology

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Outline of the Talk

- Motivations and aim of the talk
- On cosmology and classical matter
- Quantum Fields and Cosmology
- A genuine quantum effect in cosmology

Based on

- C.D., Nicola Pinamonti, V. Moretti 0812.4033 [gr-qc]
- C.D., Nicola Pinamonti, V. Moretti Comm. Math. Phys. **285** (2009), 1129
- C.D., Klaus Fredenhagen, Nicola Pinamonti, Phys. Rev. D**77** (2008) 104015

Motivations - What we know - I

The 20^{th} century thought us a few good lessons:

1) Description of interactions leads to quantum field theory on **flat spacetime**:

- it works almost perfectly for free and electroweak forces,
- it leads to the notion of elementary particles,
- perturbative QFT, renormalization, etc...

What are the underlying assumptions?

a) the form of the interaction (*i.e.*, gauge theories, self-interaction),

b) covariance under the Poincaré group.

b) together with positivity of energy entails

- a complete classification of free fields and of their dynamic,
- existence and uniqueness of a ground state for the quantum theory.

Motivations - What we know - II

2) Description of the gravitational interaction leads to **General Relativity**:

- the information on the geometry of the spacetime is encoded in the metric $g_{\mu\nu}$,
- the information of matter is encoded in the stress-energy tensor $T_{\mu\nu}$.

They **mutually influence each other** via

$$R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu}.$$

Do we understand it?

- if $T_{\mu\nu} = 0 \longrightarrow$ yes (Minkowski, Schwarzschild, de Sitter, anti de Sitter),
- if $T_{\mu\nu}$ describes classical matter such as a perfect fluid $\rightarrow$ more or less
- if not, ... well $\longrightarrow$ not really. Interpretation problems?!

Motivations - What we know - III

What is a good $T_{\mu\nu}$? Are all O.K. or should we discard some?

1. Bianchi identities lead to $\nabla^\mu T_{\mu\nu} = 0$... conservation of the stress-energy tensor (not much, but it will be later a problem!)

Classically one seeks further restricting conditions leading to

- strong energy condition $\longrightarrow$ $T_{\mu\nu}\zeta^\mu$ is future-directed timelike or null $\forall \zeta^\mu$ timelike

Problem: For classical matter it is fine, but, for interacting classical field theory ($\lambda\phi^4$ as an example), it fails. For quantum fields, also free, it always fails!

Aim of the talk - I

What are the possible solutions?

Step 1 - Consider a fixed $g_{\mu\nu}$ and try to study interactions in this framework.

This leads to **QFT on curved backgrounds**! Very difficult because:

- no symmetry is a priori present! Hence no notion of elementary particle!
- no a priori notion of positive energy; hence what is a good ground state?

After 25 years we learned a lot:

1. in presence of (Dirac) fields, there is a class of distinguished backgrounds, the globally hyperbolic ones. These are spacetimes such that $M = \Sigma \times \mathbb{R}$, *i.e.*, there exists a global temporal function,
2. there is a good alternative notion of positive energy (microlocal spectrum condition) checking the singularities of the two-point function,

Aim of the talk - II

3. one can define Wick polynomials and renormalise the theory (perturbation theory is O.K. and quantum states play no role!),
4. under sufficient hypothesis, it also exists a unique preferred ground state[1].

End-point: We can adopt a new principle of general local covariance

Step 2 - Try to approach Einstein's equations semiclassically

$$R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} = 8\pi \langle : T_{\mu\nu} : \rangle_\omega .$$

Very complicated to find solutions...

- there are scenarios in which it becomes simpler, namely Cosmology.
- for once since long, we combine mathematics, theory and experiments!
- we can answer the question: are quantum effect really important?

[1]C.D., Valter Moretti, Nicola Pinamonti: Rev. Math. Phys. **18** (2006) 349 and arXiv:0812.4033 [gr-qc]

The Cosmological Principle and FRW - I

We wish to model the geometry of the Universe!

Occam's razor leads to the Cosmological principle, *i.e.*,

- spacetime is homogeneous $\rightarrow$ "at each instant of time, all space points look the same",
- spacetime is isotropic $\rightarrow$ there is at each point an observer who sees an isotropic Universe,

This entails

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1+kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right].$$

- the parameter $k = 0, \pm 1$ tells me if spatial section are flat planes, spheres or hyperboloids,
- there is still no dynamical content. This determines $a(t)$ and, to this avail, one needs a good $T_{\mu\nu}$.

The Cosmological Principle and FRW - II

Which $T_{\mu\nu}$? Let start with *classical matter*

We know:

- part of mass-energy in the Universe is ordinary matter (stars, galaxies, clusters),
- their density is so low that they appear like “dust” with density ρ. Hence

$$T_{\mu\nu} = \rho\zeta_\mu\zeta_\nu \quad \zeta^\mu\zeta_\mu = 1$$

- if one also accounts for radiation, then there is also pressure to consider.

$$T_{\mu\nu} = \rho\zeta_\mu\zeta_\nu + P\left(g_{\mu\nu} + \zeta_\mu\zeta_\nu\right),$$

which is the stress-energy tensor of a **perfect fluid**.

The dynamic of the scale factor

I can now solve the Einstein's equation and I need only one differential equation for $a(t)$!

I assume from now on $\Lambda = 0$!

$$G_{tt} = R_{tt} - \frac{R}{2} g_{tt} = 8\pi T_{tt} \longrightarrow 3\frac{\dot{a}^2 + k}{a^2} = 8\pi\rho,$$
$$G_{xx} = R_{xx} - \frac{R}{2} g_{xx} = 8\pi T_{xx} \longrightarrow 3\frac{\ddot{a}}{a^2} = -4\pi(\rho + 3P).$$

Conservation of $T_{\mu\nu}$, *i.e.* $\nabla^\mu T_{\mu\nu} = 0$ yields

$$\dot{\rho} + 3(\rho + P)\frac{\dot{a}}{a} = 0.$$

Notice that the dynamical content boils down to this last equation and to an identity between traces:

$$Tr(G_{\mu\nu}) = -R = 8\pi Tr(T_{\mu\nu}).$$

The dynamic of the scale factor - II

To solve that system I need an **equation of state** $\rho = \gamma P$

I assume from now on $k = 0$, but only for simplicity of the talk!

Eq. of state	scale factor	conservation of $T_{\mu\nu}$
Dust, $P = 0$	$a(t) \propto t^{\frac{2}{3}}$	$\rho a^3(t) = const.$
Radiation, $P = \frac{\rho}{3}$	$a(t) \propto \sqrt{t}$	$\rho a^4(t) = const.$

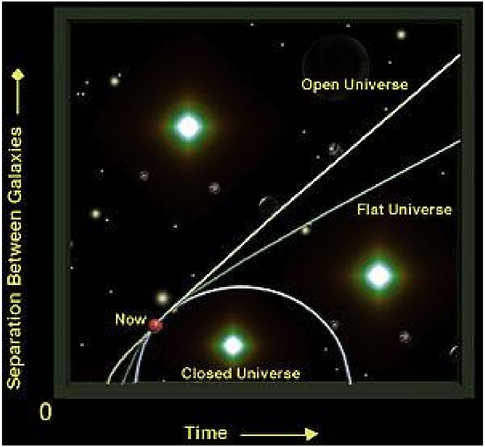
One should interpret the results, but instead let us look at the assumptions.

To get here we assumed

1. isotropy and homogeneity of the Universe,
2. the behaviour of the stress-energy tensor is classical,
3. pressure and energy density are related by an equation of state.

Are we happy?

A didactic explanation



I am bemused and bewildered

The "classical" approach to cosmology is highly unsatisfactory

on a practical ground,

- the model is far too rough in the description of matter,
- it is plagued by many problems, namely
  1. the singularity problem as $a \longrightarrow 0$, namely $\rho \longrightarrow \infty$,
  2. the flatness problem,
  3. the homogeneity problem.

and a foundational ground,

- Classical matter cannot account for an explanation of interac. as QFT.

Looking for a way out

Let us try to take Quantum Field Theory seriously.

Let us assume

1. isotropy and homogeneity of the Universe,
2. the behaviour of the stress-energy tensor is quantum,

One skips the third assumption because pressure and energy density will be automatically related by an equation of state on shell.

The big question is

Is something getting better? Do I learn something new?

My best friend: the scalar field

Let us consider what happens with a scalar field in a FRW with $k = 0$.

$$\phi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}.$$

Why?

- It is the simplest example and it has no spin structure,
- It is often used as the building block of models dealing with the early Universe (Inflation,...),
- If quantised, it contains all the essential features of a system "with Canonical Commutation Relations".

What is the dynamic of a free scalar field on a curved background?

Without Poincaré invariance, we should ask for

- the equation of motion must reduce to the standard one in Minkowski,
- the equation of motion should arise out of a Lagrangian,
- The Lagrangian should vary continuously in the parameters, namely m^2.

Classical scalar field

$$\Box_g \phi - m^2\phi - \zeta R\phi = 0$$

This comes for the following Klein-Gordon action:

$$S[\phi] = \int_{\mathbb{R}^4} d^4x \sqrt{|g|} L[\phi] = \int_{\mathbb{R}^4} d^4x \frac{\sqrt{|g|}}{2} \left(\partial^\mu \phi \partial_\mu \phi + m^2\phi^2 + \zeta R\phi^2 \right).$$

Which ζ?

- $\zeta = 0$ (minimal coupling):
  - it is the best choice from an Occam's razor perspective
  - violates symmetries under conformal rescaling

Classical scalar field - II

Notice that with $d\tau = \frac{dt}{a(t)}$

$$ds^2 = -dt^2 + a^2(t)[dr^2 + r^2 d\mathbb{S}^2(\theta,\varphi)] = a^2(\tau)[-d\tau^2 + dr^2 + r^2 d\mathbb{S}^2(\theta,\varphi)],$$

The metric is conformally the flat one! Then

- $\zeta = \frac{1}{6}$ (conformal coupling):
  - a solution ϕ_M of the KG equation in Minkowski, gives $\phi \doteq a^{-1}(t)\phi_M$,
  - if $m = 0$, the solutions fulfil Huygens' principle (supported on the light cone).
- No clearly totally favoured value for ζ.
- It exists also for Dirac fields, but there it is fixed as $\frac{1}{4}$.

Stress-energy tensor: towards quantization

$$T_{\mu\nu} = \frac{1}{\sqrt{|g|}}\frac{\delta\sqrt{|g|}L[\phi]}{\delta g_{\mu\nu}} =$$

$$= \partial_\mu\phi\partial_\nu\phi + \zeta(R_{\mu\nu} - \frac{g_{\mu\nu}}{2}R)\phi^2 - \frac{1}{2}g_{\mu\nu}L[\phi]$$

This stress-energy tensor

- is conserved *i.e.*, $\nabla^\mu T_{\mu\nu} = 0$,
- is traceless if $\zeta = \frac{1}{6}$ and $m = 0$,

$$T = -3\left(\frac{1}{6} - \zeta\right)\Box\phi^2 - m^2\phi^2.$$

What is next? Quantization

Ready and steady to quantize

What does quantization in curved backgrounds mean?

• It means a big headache,

• It means the selection of an algebra of observables containing (up to normal ordering) also $T_{\mu\nu}$,

• It means the identification of a good notion of quantum ground state.

Let us start from this last point!

Ground states in curved background

I want a state ω to make precise the notion of $n - point$ function

$$\omega(\phi(x_1)....\phi(x_n)) \doteq \langle\phi(x_1)...\phi(x_n)\rangle$$

I seek

- an ω which is quasi-free (technical condition),
- an ω which is of Hadamard form
  - they share the same ultraviolet behaviour as the ground state in Minkowski,
  - only on these states the quantum fluctuations of $T_{\mu\nu}$ are finite,

Hence in a sufficiently small neighbourhood of any point $p \in \mathbb{R}^4$, the integral kernel of the two-point function is

$$\omega(x,y) = \frac{U(x,y)}{\sigma_\epsilon(x,y)} + V(x,y)\ln\frac{\sigma_\epsilon(x,y)}{\lambda} + W(x,y)$$

A little more about Hadamard states - I

Let start again from

$$\omega(x,y) = \frac{U(x,y)}{\sigma_\epsilon(x,y)} + V(x,y)\ln\frac{\sigma_\epsilon(x,y)}{\lambda} + W(x,y)$$

One can prove that

- U, V, W are all smooth scalar functions,
- in Minkowski $U = 1$ and $V = \frac{m^2}{2\sqrt{m^2(x-y)^2}} J_1(\sqrt{m^2(x-y)^2})$, whereas in curved backgrounds they are a series

$$U(x,y) = \sum_{n=0}^{\infty} u_n(x,y)\sigma^n, \quad V(x,y) = \sum_{n=0}^{\infty} v_n(x,y)\sigma^n,$$

  which are determined out of recursion relations,
- the singular part, namely U and V, depends only on geometric quantities such as $R, R^2, R_{\mu\nu}R^{\mu\nu}\ldots$
- the choice of a quantum state of Hadamard form lies only in W

A little more about Hadamard states - II

Hadamard states entail

- I can normal order potential observables out of the building block

$$: \phi(x)\phi(y) := \phi(x)\phi(y) - H(x,y),$$

  being

$$H(x,y) = \frac{U(x,y)}{\sigma_\epsilon(x,y)} + V(x,y)\ln\frac{\sigma_\epsilon(x,y)}{\lambda}.$$

- in Minkowski it is the usual known machinery,
- I can define Wick polynomial and I can renormalise also in curved backgrounds.

Problem: (Un)fortunately anomalies arise!

The quantum stress-energy tensor has problems[2]

Out of the definition of normal ordering via Hadamard states

$$:T_{\mu\nu}:=:\partial_\mu\phi\partial_\nu\phi:+\zeta(R_{\mu\nu}-\frac{g_{\mu\nu}}{2}R):\phi^2:-\frac{1}{2}g_{\mu\nu}L[:\phi:],$$

Evaluated on an Hadamard state and it gives a finite quantity $:T_{\mu\nu}:_\omega$.

What do I require from $:T_{\mu\nu}:_\omega$?

- For consistency of the semiclassical Einstein's equations I need

$$0=\nabla^\mu(R_{\mu\nu}-\frac{R}{2}g_{\mu\nu})=\nabla^\mu:T_{\mu\nu}:_\omega\ .$$

One verifies in 1 month of calculations:

$$\nabla^\mu:T_{\mu\nu}:_\omega=cv_1(x,x)\quad c\in\mathbb{R}\setminus\{0\}$$

$$v_1(x,x)=\frac{1}{720}(R_{ij}R^{ij}-\frac{R^2}{3}+\Box R)+\frac{1}{8}(\frac{1}{6}-\zeta)^2R^2+\frac{m^4}{8}+$$
$$-\frac{1}{4}(\frac{1}{6}-\zeta)m^2R+\frac{1}{24}(\frac{1}{6}-\zeta)\Box R.$$

[2] C.D., Klaus Fredenhagen, Nicola Pinamonti, Phys. Rev. D**77** (2008)

What have we learned?

- Use a diploma/Ph.D. student to check the calculations,
- The stress-energy tensor as it stands is inadequate. Can we improve it?
- the problem is there also in Minkowski.

Idea: The classical stress-energy tensor is incomplete and I need to add something!

Solution: I can add to $T_{\mu\nu}$ a term $T_{0\mu\nu} \doteq k g_{\mu\nu} L[\phi]$ with $k \neq 0$.

- $T_{0\mu\nu}$ is classically **conserved and traceless**,
- the expectation value of $L[:\phi:]$ on ω does not vanish!
- the new term is not "coming out" of a Lagrangian.

It turns out that, if $k = \frac{1}{6}$,

$$\nabla^{\mu}(:T_{\mu\nu} + T_{0\mu\nu}:) = 0.$$

The conformal anomaly

Have I solved all problems? Yes! But I pay a price

Let us look at the trace of $T_{\mu\nu}$

$$:T:_\omega = \left(-3\left(\frac{1}{6}-\zeta\right)\Box - m^2\right)\frac{W(x,x)}{8\pi^2} + \frac{v_1(x,x)}{4\pi^2}.$$

What does it mean?

- classically if $m = 0$ and $\zeta = \frac{1}{6}$, then $T = 0$,
- at a quantum level it turns out in this case that

$$:T:_\omega = \frac{1}{4\pi^2}\left(\frac{1}{720}(R_{ij}R^{ij} - \frac{R^2}{3} + \Box R)\right)$$

- This is the conformal anomaly. It does not exist in Minkowski!
- The term $\Box R$ is *dynamically unstable* and can be mod out "renormalizing", *i.e.*,

$$L_{count} = \int_{\mathbb{R}^4} \sqrt{|g|}\left(AR^2 + BR_{\mu\nu}R^{\mu\nu}\right).$$

Effect in FRW spacetime

Recall that

$$ds^2 = -dt^2 + a^2(t)[dr^2 + r^2 d\mathbb{S}^2(\theta, \varphi)],$$

Let us take as matter a scalar field with $m \neq 0$ and $\zeta = \frac{1}{6}$. The semiclassical dynamic is encoded in

$$-R = 8\pi :T:_\omega, \quad \nabla^\mu :T_{\mu\nu}:_\omega = 0.$$

This yields with $H = \frac{\dot{a}(t)}{a(t)}$

$$-6\left(\dot{H} + 2H^2\right) = -8\pi m^2 :\phi^2:_\omega + \frac{1}{\pi}\left(-\frac{1}{30}(\dot{H}H^2 + H^4 + \frac{m^4}{4}\right).$$

One can compute that for $m^2 \gg R$ and $m^2 \gg H$, $:\phi^2:_\omega = \frac{1}{32\pi^2} m^2 + \beta R$

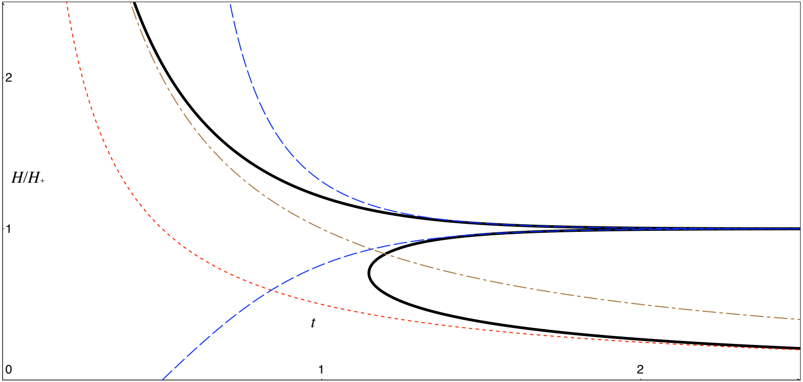
The equation for $H(t)$ becomes

$$\dot{H} = \frac{-H^4 + H_+^2 H^2}{H^2 - \frac{H_+^2}{4}},$$

$$H_+^2 = 360\pi - 2880\pi^2 m^2 \beta.$$

Endgame

Here comes the miracle! The plot:



Unveiling the future

Summary:

- There is a nice way to deal with QFT in curved backgrounds
- It has interesting properties:
  - it is rich of mathematics,
  - it is rich of physical models,
  - it can lead to **numbers** to be experimentally probed.

The big quest:

- Tackle inflation form a rigorous point of view,
- Derive testable results and not only formal theorems,
- Start a Brans-Dicke like programme for cosmology,
- Give a coherent and sound derivation of the inflation effects on the CMB spectrum (first step done!).