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Towards a Rigorous Functional Framework for Classical Field Theory: Algebraic Structure and Dynamics

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Introduction

Nowadays the framework of classical field theory most commonly employed by physicists is based on (formal) functional methods, tailored to the needs of (path-integral-based) quantum field theory.

- Heuristic infinite-dimensional generalisation of Lagrangian mechanics;
- To make it mathematically rigorous is possible, but technically quite complicated at the level of generality stated above – usually done in the (restricted) setting of Banach spaces, doing it at the (more natural) level of Fréchet spaces (e.g. smooth field configurations) precludes the use of many local analytic tools.

Common (albeit subtle) conceptual misunderstanding lies in the 1st. point. Namely, we emphasize:

> "Classical field theory is not as 'infinite dimensional' as it might appear!"

The underlying reason is locality (in spacetime).

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Several reasons to emphasize locality:

1. Euler-Lagrange equations of motion are PDE's (put in another way: variational principle is local);

2. Physical classical field theories are hyperbolic $\Rightarrow$ propagation speed of dynamical effects is finite (relativistic microcausality);

3. Locality and relativistic microcausality play a pivotal role also in rigorous approaches to quantisation.

Let's be more precise about all this. Let $(\mathcal{M}, g)$ be a globally hyperbolic Lorentzian manifold, with volume element $\mathrm{d}\mu_g = \sqrt{|\det g|}\mathrm{d}x$, $\mathcal{E} \xrightarrow{p} \mathcal{M}$ a (vector) bundle over $\mathcal{M}$ endowed with a connection ∇, i.e. a global section of the 1st order jet bundle $J^1\mathcal{E} \xrightarrow{t} \mathcal{E}$. We'll call $\Gamma^\infty(\mathcal{M}, \mathcal{E})$ a(n off-shell) field configuration space.

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Kinematics and observables of classical field theory

$\Gamma^\infty(\mathcal{M}, \mathcal{E})$ is endowed with the usual Fréchet topology. We'll single out our class of observable quantities:

Definitions

We say that a function(al) F defined on $\Gamma^\infty(\mathcal{M}, \mathcal{E})$ is:

Smooth if for all k the k-th order derivatives

$$F^{(k)}[\phi](\delta\phi^{\otimes k}) \doteq \frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}\!\upharpoonright_{\lambda=0} F(\phi + \lambda\delta\phi)$$

exist as jointly continuous maps from $\Gamma^\infty(\mathcal{M}, \mathcal{E})^{k+1}$ to $\mathbb{R}$. In particular, $F^{(k)}[\phi]$ is a distribution (density) of compact support;

Additive if for all $\phi_1, \phi_2, \phi_3 \in \Gamma^\infty(M, \mathcal{E})$ such that $\operatorname{supp}\phi_1 \cap \operatorname{supp}\phi_3 = \varnothing$, then

$$F(\phi_1 + \phi_2 + \phi_3) = F(\phi_1 + \phi_2) - F(\phi_2) + F(\phi_2 + \phi_3);$$

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Local if it's additive and, for all $\phi \in \Gamma^\infty(\mathscr{M}, \mathscr{E})$,
$WF(F^{(k)}[\phi]) \perp T\Delta^k(\mathscr{M})$, where
$\Delta^k(\mathscr{M}) \doteq \{(x, \ldots, x) \in \mathscr{M}^k : x \in \mathscr{M}\}$ is the thin diagonal of
$\mathscr{M}^k$. In particular, $F^{(1)}[\phi]$ is a smooth function for each fixed ϕ.

Some basic key results:

1. If we define the spacetime support of a functional F as

   $$\mathrm{supp}F \doteq \mathscr{M} \backslash \{x \in \mathscr{M} : \exists U \ni x \text{ open s.t. } \forall \phi, \psi, \mathrm{supp}\phi \subset U, F(\phi+\psi) = F(\psi)\},$$

   additivity implies that any functional with compact spacetime support can be decomposed as a finite sum of functionals of arbitrarily small support.

2. For smooth, spacetime compactly supported functionals, additivity also entails that $\mathrm{supp}F^{(k)}[\phi] \subset \Delta^k(\mathscr{M})$, and locality, that F must be of the form

   $$F(\phi) = \int_{\mathscr{M}} j^m \phi^* \mathscr{L}(x) \mathrm{d}\mu_g(x),$$

   with $\mathscr{L} \in \mathscr{C}_c^\infty(J^m \mathscr{E})$ for some m.

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- The idea that additivity should characterise locality for nonlinear functionals goes back to [Kantorovitch-Pinsker '38, '39], in the context of generalised random processes, and it has arisen occasionally in probability [Gel'fand–Vilenkin '64, Rao '71, '80] and in the study of certain nonlinear integral equations [Chacón–Friedman '65, Krasonel'skii '65, Friedman-Katz '69], but has remained unknown in classical field theory.

- Recent developments regarding the renormalisation group in perturbative algebraic quantum field theory [Brunetti–Dütsch–Fredenhagen '09, arXiv:0901.2038; Brunetti–Fredenhagen '09, arXiv:0901.2063] have singled out the class of smooth, compactly spacetime supported local functionals (notation: $\mathfrak{F}_{loc}(\mathcal{M}, \mathscr{E})$) as the most relevant one for classical field theory.

- $\mathfrak{F}_{loc}(\mathcal{M}, \mathscr{E})$, however, is not closed under (pointwise) products. A slightly larger class is given by the microcausal functionals

$$\mathfrak{F}_{mc}(\mathcal{M}, \mathscr{E}) = \{F : \Gamma^\infty(\mathcal{M}, \mathscr{E}) \to \mathbb{R} \text{ smooth: } \mathrm{supp}F \text{ compact,}$$

$$WF(F^{(k)}[\phi]) \cap ((\mathcal{M} \times (J^+(0)))^k \cup (\mathcal{M} \times (J^-(0)))^k) = \varnothing, \forall \phi\},$$

which is closed under products. Moreover, the latter is also closed under Poisson brackets (to be defined later).

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Local dynamics and Main Result

Typical elements of $\mathfrak{F}_{loc}(\mathcal{M}, \mathcal{E})$ are the action functionals

$$S(f)[\phi] = \int_{\mathcal{M}} f(x) j^{m-1} \phi^* \mathcal{L}(x) \mathrm{d}\mu_g(x), \ f \in \mathcal{C}_c^\infty(\mathcal{M}).$$

In the remainder of the talk, we shall set $m = 2$ and be interested in the following problem: consider for concreteness...

- A scalar field $\phi \in \Gamma^\infty(\mathcal{M}, \mathcal{E}) \doteq \mathcal{C}^\infty(\mathcal{M})$ in a globally hyperbolic spacetime $(\mathcal{M}, g)$, and
- Two (1st-order) action functionals

$$S_i(f)[\phi] = \int_{\mathcal{M}} f(x) \mathcal{L}_i(x, \phi(x), \partial^1 \phi(x)) \mathrm{d}\mu_g(x), \ i = 1, 2$$

with (strictly hyperbolic) Euler-Lagrange derivatives

$$S_{i(1)}[\phi] = \nabla_a \frac{\partial \mathcal{L}_i}{\partial \nabla_a \phi} - \frac{\partial \mathcal{L}_i}{\partial \phi},$$

such that

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i.)

$$S_1 - S_2 = F(\lambda h) = \lambda \int_{\mathscr{M}} \sqrt{|\det g(x)|} \mathrm{d}x \mathscr{L}_{int}(x, \phi(x), \partial^1 \phi(x), \lambda h(x))$$

with $h \in \mathscr{C}_c^\infty(\mathscr{M})$ and $\mathrm{supp} j^1 \phi^* \mathscr{L}_{int}(., h(.)) \subset \mathrm{supp} h$ (interaction term with "spacetime-cutoff" coupling constant), $\lambda > 0$, and

ii.) $F(\lambda h)_{(1)}[\phi]$ depends pointwise on ϕ and at most its first derivatives $\nabla \phi$ (we shall discuss later how one can drop this restriction).

We want to

Main Goal & Definition

Prove the existence of a map $r_{S_1,S_2} : \mathscr{C}^\infty(\mathscr{V}) \to \mathscr{C}^\infty(\mathscr{V})$ in an open neighbourhood $\mathscr{V}$ of $\mathrm{supp} h$ such that

$$S_{1(1)} \circ r_{S_1,S_2} = S_{2(1)}, \tag{1}$$

$$r_{S_1,S_2}(\phi)(x) = \phi(x), x \notin J^+(\mathrm{supp} h). \tag{2}$$

We call r_{S_1,S_2} the retarded Møller operator of S_1 w.r.t. S_2.

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- r_{S_1, S_2} appears naturally in the context of perturbative algebraic QFT (Dütsch–Fredenhagen CMP'03, Brunetti–Fredenhagen *ibid*., Brunetti–Dütsch–Fredenhagen *ibid*.), where h plays both the role of an IR regulator and of a localization for the algebra of perturbative interacting fields.

- When acting on solutions of $S_{2(1)}[\phi] = 0$ r_{S_1, S_2} can be seen as an intertwiner of (on-shell) covariant phase spaces or, equivalently, as the solution of a "covariant" Cauchy problem.

- (1)–(2) also mean that $r_{S_1, S_2}(\phi)$ solves an inhomogeneous (off-shell) nonlinear hyperbolic PDE with prescribed initial conditions in the past of $\mathrm{supp}\, h \Rightarrow$ very few rigorous well-posedness results exist, qualitative behaviour of solutions can be dramatically changed – (parabolic) example: (incompressible) Navier-Stokes equations

$$\underbrace{\partial_t \mathbf{v} - \nu \triangle \mathbf{v}}_{\text{free (heat) part}} + \underbrace{(\mathbf{v}.\nabla)\mathbf{v}}_{\text{interaction}} = \underbrace{-\nabla p}_{\text{source}} \rightarrow \left\{ \begin{array}{l} \nabla p = 0 : \text{only laminar flow} \\ \nabla p \neq 0 : \text{turbulence!} \end{array} \right.$$

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Coupling as an off-shell flow parameter $\Rightarrow$ Main Claim

- It's clear that r_{S_1,S_2} exist on shell whenever local well posedness for $S_{1(1)}[\psi] = (S + F(\lambda h))_{(1)}[\psi] = 0$ in a ngb. of supp$h$ holds.

- Concerning *global* existence (i.e., in the whole of the spacetime $\mathscr{M}$), in the future of supp$h$ (1) tells us that, under the further hypothesis

  iii.) $S_2 \doteq S$ is quadratic ("free");

  $\psi = r_{S_1,S_2}(\phi) - \phi$ solves $S_{(1)}[\psi] = 0 \Rightarrow$ finding r_{S_1,S_2} globally boils down to finding it locally!

- As for the latter problem...

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> **Caveat 1**
>
> Notice that, under the hypotheses i.)–iii.), we can use (2) together with the linearity of $S_{(1)}$ and apply the (background-independent) retarded fundamental solution Δ_S^R of $S^{(1)} = S^{(2)}[\psi], \forall \psi$ to
>
> $$S_{(1)}[r_{S+F(\lambda h),S}(\phi) - \phi] = F(\lambda h)[r_{S+F(\lambda h),S}(\phi)]$$
>
> and obtain the so-called Yang-Feldman equation
>
> $$r_{S+\lambda F(h),S}(\phi) = \phi - \lambda \Delta_S^R \circ F(h)_{(1)}[r_{S+\lambda F(h),S}(\phi)].$$

- More generally, if $S_{(1)}$ is a semilinear operator with a compactly supported nonlinearity, say $S_{int,(1)}$, one can add the term $S_{int,(1)}[\phi] - S_{int,(1)}[r_{S+F(\lambda h),S}(\phi)]$ to the right-hand side of the Yang-Feldman equation. In either case, solving the latter was a task essentially accomplished already in the 70's (Segal, Morawetz, Strauss) by using energy estimates and a Duhamel iteration scheme.

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- We shall be concerned with local existence of $r_{S+F(\lambda h),S}$ when no such restrictions on S are imposed. In that case, the Yang-Feldman strategy simply breaks down, and one needs a new strategy, as follows.

- Differentiating (1) w.r.t. λ leads to

$$(S_{(1)} + \lambda F(h)_{(1)})^{(1)}[r_{S+F(\lambda h),S}(\phi)] \circ \frac{\mathrm{d}}{\mathrm{d}\lambda} r_{S+F(\lambda h),S}(\phi)$$

$$+ \bar{F}_{(1)}(\lambda h)[r_{S+F(\lambda h),S}(\phi)] = 0, \text{ where } \bar{F}_{(1)}(\lambda h) \doteq h\frac{\delta}{\delta h'}F_{(1)}(h' = \lambda h). \quad (3)$$

- Now we invoke (2) and apply the retarded fundamental solution $\Delta^R_{S+F(\lambda h)}[r_{S+F(\lambda h),S}(\phi)]$ of the linearised Euler-Lagrange operator $(S + F(\lambda h))^{(1)}_{(1)}[r_{S+F(\lambda h),S}(\phi)]$ around the background $r_{S+F(\lambda h),S}(\phi)$ to the left of both sides of (3)

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} r_{S+F(\lambda h),S}(\phi) = -\Delta^R_{S+F(\lambda h)}[r_{S+F(\lambda h),S}(\phi)] \circ \bar{F}_{(1)}(\lambda h)[r_{S+F(\lambda h),S}(\phi)], \quad (4)$$

which shows that $\psi(\lambda) \doteq r_{S+F(\lambda h),S}(\phi)$ is the unique solution of the flow equation (4) with initial condition $\psi(0) = \phi$.

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- Formally integrating (4) w.r.t. λ on both sides and using the initial condition above, we arrive at

$$r_{S+F(\lambda h),S}(\phi) = \phi - \int_0^\lambda \mathrm{d}\lambda' \Delta^R_{S+F(\lambda' h)}[r_{S+F(\lambda' h),S}(\phi)] \circ \bar{F}_{(1)}(\lambda' h)[r_{S+F(\lambda' h),S}(\phi)].$$
(5)

- We could keep proceeding formally by iterating (4) and write $r_{S+\lambda F(h),S}(\phi)$ as a formal power series of its composition with local functionals G [Dütsch–Fredenhagen *ibid.*]

$$G \circ r_{S+F(\lambda h),S} \sim \sum_{k=0}^\infty \frac{\lambda^k}{k!} \frac{\mathrm{d}^k}{\mathrm{d}\lambda^k} \big\rvert_{\lambda=0} G \circ r_{S+F(\lambda h),S}.$$

This is actually the method used in (perturbative algebraic) quantum field theory, barring renormalisation issues.

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- However, our aim is nonperturbative, and thus achieved by looking at the map

$$\psi(\lambda) \mapsto \phi(\lambda) = \psi(\lambda) + \int_0^\lambda \mathrm{d}\lambda' \Delta^R_{S+F(\lambda h)}[\psi(\lambda')] \circ \bar{F}_{(1)}(\lambda' h)[\psi(\lambda')], \quad (6)$$

  which just defines the inverse $r^{-1}_{S+F(\lambda h),S}$ of $r_{S+F(\lambda h),S}$.
- From now on, for the sake of pedagogy we shall set $(\mathscr{M}, g) = \mathbb{R}^{1,d-1} \ni (x^0 = t, x)$.

Main Claim

The map (6) is invertible in a neighbourhood of zero in $\mathscr{C}^1([0,\Lambda], \mathscr{C}^\infty(\mathscr{M}))$; its inverse satisfies (1), (2).

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Towards a proof of Main Claim

- The crucial step in our proof is to obtain a priori estimates on $\Delta^R_{S+F(\lambda h)}[\psi]$ in terms of both the linear and the nonlinear (background) arguments. These are essentially refined energy estimates for $S^{(1)}_{(1)} + \lambda \bar{F}(\lambda h)^{(1)}_{(1)}$ which state explicitly their dependence on the latter's coefficients [Klainerman '78–'80–'82]. We shall formulate a stronger, local form of these inequalities, within a more geometric formulation [Klainerman '01].

- Suppose that there exist $0 < T$ and a point q such that $q^0 > T$ and such that supp$h$ is contained in the interior of the truncated cone $D_{0,T}(q) \doteq \{(t,x) \in J^-(q) : 0 \leq t \leq T\}$ with level sets $\Sigma_t(q) \doteq \{(t,x) \in J^-(q)\}$, and define the local energy norms

$$\|\psi\|_{E^k,q} \doteq \sup_{t' \in [0,T]} \|\psi(t',.)\|_{H^{(k+1)}_x(\Sigma_{t'}(q))} + \sup_{t' \in [0,T]} \|\partial_t \psi(t',.)\|_{H^{(k)}_x(\Sigma_{t'}(q))}.$$

- The Proposition below shows, among other things, that $r_{S+F(\lambda h),S}(\phi) - \phi$ is not a Lipschitz map with respect to any $\|.\|_{E^k,q}$, (PDE folk wisdom: "hyperbolic equations are not strongly stable w.r.t. perturbations of the coefficients." [Tataru ICM '02])

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Proposition (refined local energy estimates)

For $\phi, \delta\phi \in E^\infty(q) \doteq \{\psi : \|\psi\|_{E^k,q} < +\infty, \forall k \geq 0\}$ we have

$$\|\Delta^R_{S+F(\lambda h)}[\phi]\delta\phi\|_{E^0,q} \leq D_0 \sup_{t' \in [0,T]} \|\delta\phi\|_{L^2_x(\Sigma_{t'}(q))}, \qquad (7)$$

$$\|\Delta^R_{S+F(\lambda h)}[\phi]\delta\phi\|_{E^k,q} \leq D_k \left(\|\delta\phi\|_{E^{k-1},q} + \sup_{t' \in [0,T]} |(\bar{F}^{(1)}_{(1)})(t',.)|_{\mathscr{C}^k_x(\Sigma_{t'}(q))} \|\delta\phi\|_{E^0,q} \right),$$
$$(8)$$

where D_k, $k \geq 0$ are constants which depend only on d, T and $\|\phi\|_{\mathscr{C}^1(\text{supp}h)}$.

- Applying Sobolev inequalities and Schauder estimates to the spatial $\mathscr{C}^k$ norms of $(\bar{F}^{(1)}_{(1)})(t',.)$ in (8), we arrive at

$$\|\Delta^R_{S+F(\lambda h)}[\phi]\delta\phi\|_{E^k,q} \leq D'_k [\|\delta\phi\|_{E^{k-1},q} + (1 + \|\phi\|_{E^{k+1+[\frac{d+1}{2}]},q}) \|\delta\phi\|_{E^0,q}],$$
$$(9)$$

where $[s]$ gives the integer part of s.

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Nash–Moser–Hörmander iteration scheme

- The argument above shows that one loses $1 + \left[\frac{d+1}{2}\right]$ derivatives at each iteration when trying to solve (5) by a fixed-point method (see Caveat 2!). This phenomenon has no on-shell counterpart.

- Alternative: combine fixed-point method with a "multiscale" (Paley-Littlewood) smoothing procedure that is gradually removed at each iteration $\Rightarrow$ the former doesn't converge fast enough to give overall convergence for the above loss of derivatives, but using a Newton iteration scheme instead does. The result is the celebrated

Theorem (Nash–Moser–Hörmander)

Let $\Phi : \mathcal{U} \subseteq E^\infty(q) \cap \{\psi : \|\psi - \psi_0\|_{E^\mu,q} < R\} \to E^\infty(q)$, $\mu \in \overline{\mathbb{Z}}_+$, $R > 0$ be twice Gâteaux differentiable satisfying for all $k \geq 0$ the tame estimates

$$\|\Phi(\psi)\|_{E^k,q} \leq C_k (1 + \|\psi\|_{E^{k+r_0},q}) \text{ for some } r_0 > 0, \tag{10}$$

$$\|\Phi'(\psi)(\delta\psi)\|_{E^k,q} \leq C'_k[(1+\|\psi\|_{E^{k+r_1},q})\|\delta\psi\|_{E^{s_1},q}+\|\delta\psi\|_{E^{k+s_1},q}] \text{ for some } r_1, s_1 > 0, \tag{11}$$

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$$\|\Phi''(\psi)(\delta_1\psi, \delta_2\psi)\|_{E^k,q} \leq C_k''[(1 + \|\psi\|_{E^{k+r_2},q})\|\delta_1\psi\|_{E^{s_2},q}\|\delta_2\psi\|_{E^{t_2},q}$$

$$+ \|\delta_1\psi\|_{E^{s_2},q}\|\delta_2\psi\|_{E^{k+t_2},q} + \|\delta_1\psi\|_{E^{k+t_2},q}\|\delta_2\psi\|_{E^{s_2},q}], \text{ for some } r_2, s_2, t_2 > 0,$$

$$\tag{12}$$

and such that for all ψ in $\mathcal{V} \subset \{\psi : \|\psi - \psi_0\|_{E^{\mu'},q} < R'\}$, $\mu' \in \bar{\mathbb{Z}}_+$, $R' > 0$ there is a right inverse $\Psi(\psi)$ to $\Phi'(\psi)$ w.r.t. the linear factor satisfying for all $k \geq 0$ the tame estimates

$$\|\Psi'(\psi)(\delta\psi)\|_{E^k,q} \leq C_k'''[(1 + \|\psi\|_{E^{k+a_1},q})\|\delta\psi\|_{E^{b_1},q} + \|\delta\psi\|_{E^{k+b_1},q}]$$

$$\text{for some } a_1, b_1 > 0. \tag{13}$$

Then, for all k sufficiently large, there is a $R_k > 0$ such that for all $\phi \in E^\infty(q)$ fulfilling $\|\phi\|_{E^{k+b_1},q} < R_k$ the equation $\Phi(\psi) = \Phi(\psi_0) + \phi$ has a unique solution $\psi = \psi(\phi)$ such that $\|\psi(\phi) - \psi_0\|_{E^k,q} \leq R''\|\phi\|_{E^{k+b_1},q}$. In particular, if ϕ also belongs to $E^\infty(q)$, so does $\psi(\phi)$.

In our problem, we take $\psi_0 \equiv 0$ and add a dependence in λ.

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Tame (Gâteaux) differentiability of $\Delta^R_{S+\lambda F(h)}[\psi]$

To check that Φ_λ fulfills the hypotheses of the Theorem, first we collect some "resolvent-type" formulae coming from the definition of a fundamental solution, hereby stated in the case when F depends linearly on λh [Dütsch–Fredenhagen *ibid.*]

$$\Delta^{R(1)}_{S+F(\lambda h)}[\psi](\delta\psi) = -\Delta^R_{S+F(\lambda h)}[\psi] \circ F(h)^{(2)}_{(1)}[\psi](\delta\psi, \Delta^R_{S+F(\lambda h)}[\psi]), \qquad (14)$$

$$\frac{d}{d\lambda}\Delta^R_{S+\lambda F(h)}[\psi] = -\Delta^R_{S+F(\lambda h)}[\psi] \circ F(h)^{(1)}_{(1)}[\psi] \circ \Delta^R_{S+F(\lambda h)}[\psi], \qquad (15)$$

$$\Delta^{R(2)}_{S+F(\lambda h)}[\psi](\delta_1\psi, \delta_2\psi) =$$
$$= \Delta^R_{S+F(\lambda h)}[\psi] \circ F(h)^{(2)}_{(1)}(\delta_1\psi, \Delta^R_{S+F(\lambda h)}[\psi]) \circ F(h)^{(2)}_{(1)}(\delta_2\psi, \Delta^R_{S+F(\lambda h)}[\psi]) +$$
$$+ \Delta^R_{S+F(\lambda h)}[\psi] \circ F(h)^{(2)}_{(1)}(\delta_2\psi, \Delta^R_{S+F(\lambda h)}[\psi]) \circ F(h)^{(2)}_{(1)}(\delta_1\psi, \Delta^R_{S+\lambda F(h)}[\psi]) +$$
$$- \Delta^R_{S+F(\lambda h)}[\psi] \circ F(h)^{(3)}_{(1)}(\delta_1\psi, \delta_2\psi, \Delta^R_{S+F(\lambda h)}[\psi]). \qquad (16)$$

Equation (15) shows in particular that $\Delta^R_{S+F(\lambda h)}[\psi]$ is strongly differentiable (hence strongly continuous) in λ, thus allowing all the computations we need.

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Tame estimates for iteration map, end of proof

From (14) and (16), one get the following formulae for the first two derivatives of the iteration map Φ_λ (6):

$$\Phi'_\lambda(\psi)(\delta\psi) = \delta\psi +$$

$$+ \int_0^\lambda \mathrm{d}\lambda' \left(\Delta^R_{S+F(\lambda'h)}[\psi] \circ F(h)^{(1)}_{(1)}[\psi](\delta\psi) + \Delta^{R(1)}_{S+F(\lambda'h)}[\psi](\delta\psi) \circ F(h)_{(1)}[\psi] \right),$$

$$\tag{17}$$

$$\Phi''_\lambda(\psi)(\delta_1\psi, \delta_2\psi) = \int_0^\lambda \mathrm{d}\lambda' \left(\Delta^R_{S+F(\lambda'h)}[\psi] \circ F(h)^{(2)}_{(1)}[\phi](\delta_1\phi, \delta_2\phi) + \right.$$

$$+ \Delta^{R(1)}_{S+F(\lambda'h)}[\psi](\delta_1\psi) \circ F(h)^{(1)}_{(1)}[\phi](\delta_2\phi) + \Delta^{R(1)}_{S+F(\lambda'h)}[\psi](\delta_2\psi) \circ F(h)^{(1)}_{(1)}[\phi](\delta_1\phi) +$$

$$\left. + \Delta^{R(2)}_{S+F(\lambda'h)}[\psi](\delta_1\psi, \delta_2\psi) \circ F(h)_{(1)}[\phi] \right),$$

$$\tag{18}$$

where $\Delta^{R(1)}_{S+F(\lambda'h)}[\psi]$ and $\Delta^{R(2)}_{S+F(\lambda'h)}[\psi]$ are respectively given by (14) and (16). Notice that $\frac{\mathrm{d}}{\mathrm{d}\lambda}\Phi_\lambda(\psi)$, seen as a linear map acting on $\delta\psi$ for fixed ψ, doesn't lose derivatives in the semilinear case, due to the fact that the assumed loss in $F(h)_{(1)}$ is exactly compensated by the smoothing effect of $\Delta^R_{S+F(\lambda'h)}[\psi]$.

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End of proof

- The Proposition, together with Schauder estimates, show that Φ_λ satisfy the tame estimate (10) with $a_0 = [\frac{d+1}{2}] + 1$ for $\sup_{\lambda' \in [0,\lambda]} \|\psi(\lambda')\|_{E^{[\frac{d+1}{2}]+1},q} < R$, that is, $\mu = [\frac{d+1}{2}] + 1$.

- Formulae (17)–(18) show that $\Phi'_\lambda(\psi)(\delta\psi)$ and $\Phi''_\lambda(\psi)(\delta_1\psi, \delta_2\psi)$ fulfill resp. the tame estimates (11) and (12) with $r_1 = r_2 = [\frac{d+1}{2}] + 1$ and $s_1 = s_2 = t_2 = 1$.

- Finally, due to (15) and the remark following (18), $\frac{\mathrm{d}}{\mathrm{d}\lambda}\Phi'_\lambda(\psi)$ is a bounded and uniformly strongly continuous (in λ) linear map $\Rightarrow \Phi'_\lambda(\psi)$ be inverted in the semilinear case by means of a Dyson series. Iterating the tame estimate for $\frac{\mathrm{d}}{\mathrm{d}\lambda}\Phi'_\lambda(\psi)$, together with the argument for the convergence for the Dyson series, leads to the tame estimate (13) with $a_1 = [\frac{d+1}{2}] + 1$, $b_1 = 1$ and $\mu' = [\frac{d+1}{2}] + 2$ for the right inverse.

- Now... Just plug in the data above, run the "Nash–Moser–Hörmander machine", and we get local existence and uniqueness of $r_{S+F(\lambda h),S}$ in $E^\infty(q)$. The intertwining relation (1) shows that actually $r_{S+F(\lambda h),S}(\phi) \in \mathscr{C}^\infty$ for $\phi \in \mathscr{C}^\infty$. $\qquad \square$

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Structural consequences

The existence and properties of $r_{S+F(\lambda h),S}$ have fundamental implications for the underlying Poisson structure of any classical field theory determined by an action functional S, given by the Peierls bracket

$$\{F, G\}_S \doteq F_{(1)}[.] \circ (\Delta_S^R[.] - \Delta_S^A[.]) \circ G_{(1)}[.]$$

of microcausal functionals F, G. Here $\Delta_S^A[\psi]$ is the advanced fundamental solution of $S_{(1)}^{(1)}[\psi]$ around the background ψ, which is simply the adjoint of $\Delta_S^R[\psi]$.

Corollary $r_{S+F(\lambda h),S}$ is a canonical transformation, i.e. it intertwines the Poisson structures associated to S and $S + F(\lambda h)$:

$$\{.,.\}_{S+\lambda F(h)} \circ r_{S+\lambda F(h)} = \{.,.\}_S.$$

In particular, even off shell does it allow one to put $\{.,.\}_{S+F(\lambda h)}$ in "normal form", i.e. to make it locally background-independent ("Functional Darboux Theorem").

Scholium The space of microcausal functionals vanishing on solutions of $(S + F(\lambda h))_{(1)}[\psi] = 0$ is a Poisson ideal.

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Coda: final considerations

- We've given a characterisation of the classes of functionals of off-shell field configurations $\mathfrak{F}_{loc}(\mathscr{M}, \mathscr{E})$ and $\mathfrak{F}_{mc}(\mathscr{M}, \mathscr{E})$, relevant for classical field theory. It can be further shown [Brunetti–Fredenhagen–PLR, work in progress] that $\mathfrak{F}_{mc}$ carries the structure of a nuclear topological Poisson algebra for any given dynamics, in a way amenable to quantisation and renormalisation.

- We've shown the existence of r_{S_1,S_2} for "sufficiently small" field configurations around a given one. This latter condition can be controlled in general by adjusting λ (coupling strength) or supp$h$ (lifespan).

- If the Cauchy problem for $S_{1(1)}[\psi] = 0$ is well-posed in the large, one can use (2) and the composition property of r_{S_1,S_2}

$$r_{S,S} = \mathbb{1}, r_{S_2,S_3} \circ r_{S_1,S_2} = r_{S_1,S_3}$$

stemming from (1) to remove the cutoff (i.e. dependence on h) $\Rightarrow$ probably impossible off shell, unless probably in a suitable algebraic sense ("algebraic adiabatic limit" – [Brunetti–Dütsch–Fredenhagen *ibid*.]).

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- The composition property above can be combined with the support decomposition theorem for local functionals so as to provide a construction of r_{S_1, S_2} on any globally hyperbolic spacetime; only the control of the extra error terms due to curvature and the absence of Killing fields is more cumbersome but can be made systematic thanks to the geometric methods in [Klainerman '01].

- We illustrated our strategy for the case of a scalar field, but the argument carries through for sections of arbitrary vector bundles.

- Alas, the more general quasilinear case (e.g. general relativity) seems to pose some new difficulties; the Dyson series argument to invert Φ'_λ w.r.t. the linear factor fails since one then loses one derivative at each order. One can, though, circumvent this issue by means of paradifferential calculus [Bahouri-Chemin '99, Klainerman '01].