Looking at the Sky with the eyes of algebraic quantum field theory

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General Framework

We have strong evidences that

1) Electroweak and Strong Interactions $\longrightarrow$ quantum field theory on **flat spacetime**

2) Gravitational interaction $\longrightarrow$ **General Relativity**

3) Algebraic approach, $\longrightarrow$ natural platform for QFT on curved backgrounds [Brunetti, Dimock, Fredenhagen, Hollands, Kay, Moretti, Wald, Verch,...]

Are my convictions faltering when I look up at the Sky?

- Cosmology is the natural playground to test the algebraic approach to QFT on curved spacetimes

Motivations - Textbook Cosmology

Modern Cosmology has some remarkable aspects:

- Classical Cosmology is modelled by rather simple solutions of Einstein's equations. Assumptions are:
  1. isotropy and homogeneity of the Universe,

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1+kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right], \quad k = 0, \pm 1$$

  2. the stress-energy tensor is classical (perfect fluid)

$$T_{\mu\nu} = \rho\zeta_\mu\zeta_\nu + P\left(g_{\mu\nu} + \zeta_\mu\zeta_\nu\right), \quad \zeta^\mu\zeta_\mu = 1$$

  3. pressure P and energy density ρ are related by an equation of state

$$P = \gamma\rho$$

- it is plagued by some interpretation problems (homogeneity, flatness, the singularity problem...)

Motivations - “Modern” Perspective

The quest to solve those problems prompted

- Modern Cosmology $\longrightarrow$ matter is often modelled by a scalar field
- Bright side
  1. it takes seriously the role of QFT as the natural playground to discuss (quantum) matter and interactions,
  2. it provides a nice exit to most of the problems of standard cosmology,
  3. models of inflation lead to testable consequences, on the temperature of CMB in particular.
- Dark side
  1. still plenty of open theoretical problems (dark matter, dark energy...),
  2. it is unclear how to derive the used models from “first principles”,
  3. unclear concepts in curved backgrounds: (temperature)...

Can AQFT help to have a less blurred image of the Cosmo?

A semiclassical effect - I

Let us look at our framework:

- We fix the background as an FRW spacetime with flat spatial section

$$ds^2 = -dt^2 + a^2(t)[dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)], \quad M \equiv \mathbb{R} \times \mathbb{R}^3$$

- we consider for "simplicity of the talk" a scalar field on M

$$\left(\Box_g - \frac{R}{6} - m^2\right)\phi(x) = 0,$$

  which is conformally coupled to scalar curvature.

- we shall seek solutions of $G_{\mu\nu} = 8\pi\langle : T_{\mu\nu} : \rangle_\omega$, $\longrightarrow$ in FRW

$$-R = 8\pi\langle : T : \rangle_\omega$$

Intermezzo: the quest for an Hadamard state

What is a good choice for ω?

A physically reasonable choice is

1. an ω which is quasi-free (technical condition)
2. an ω which is of Hadamard form
   - same ultraviolet behaviour as the ground state in Minkowski,
   - only on these states the quantum fluctuations of $T_{\mu\nu}$ are finite,

Hence in a geodesic normal neighbourhood of any point $p \in \mathbb{R} \times \mathbb{R}^3$, the integral kernel of the two-point function is

$$\omega(x,y) = \frac{U(x,y)}{\sigma_\epsilon(x,y)} + V(x,y) \ln \frac{\sigma_\epsilon(x,y)}{\lambda} + W(x,y)$$

Intermezzo - II

Let start again from

$$\omega(x,y) = H(x,y) + W(x,y) = \frac{U(x,y)}{\sigma_\epsilon(x,y)} + V(x,y)\ln\frac{\sigma_\epsilon(x,y)}{\lambda} + W(x,y)$$

One can prove that

- U, V, W are all smooth scalar functions,
- they are expressed as an asymptotic series

$$U(x,y) = \sum_{n=0}^{\infty} u_n(x,y)\sigma^n, \quad V(x,y) = \sum_{n=0}^{\infty} v_n(x,y)\sigma^n,$$

  determined out of recursion relations,
- the singular part, namely U and V, depends only on geometric quantities such as $R, R^2, R_{\mu\nu}R^{\mu\nu}\ldots$
- the choice of a quantum state of Hadamard form lies only in W,
- we can regularize as : $\phi^2(x) := \lim_{x\to y}(\phi(x)\phi(y) - H(x,y))$.

A semiclassical effect - II

Let us *assume* to take an Hadamard state! Then

$$\langle : T : \rangle_\omega = -m^2 \frac{W(x,x)}{8\pi^2} + \frac{v_1(x,x)}{4\pi^2},$$
$$v_1(x,x) = \frac{1}{720}\left(R_{ij}R^{ij} - \frac{R^2}{3} + \Box R\right) + \frac{m^4}{8}.$$

This is the conformal anomaly. Its effect on the dynamical system $\rightarrow$

$$-6\left(\dot{H} + 2H^2\right) = -8\pi m^2 \langle : \phi^2 : \rangle_\omega + \frac{1}{\pi}\left(-\frac{1}{30}(\dot{H}H^2 + H^4) + \frac{m^4}{4}\right),$$

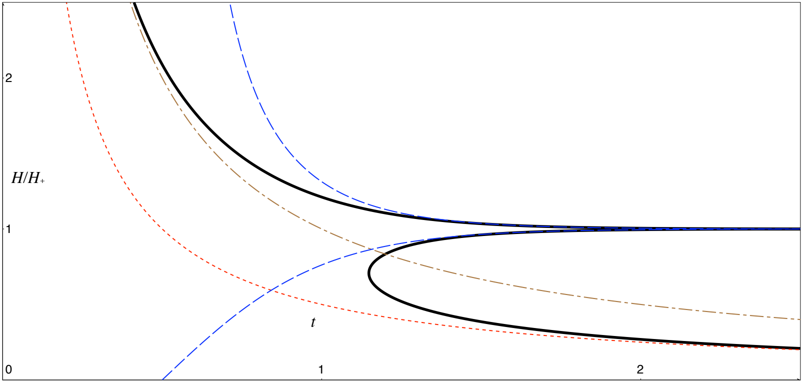
where $H = \frac{\dot{a}(t)}{a(t)}$.

A notion of approximate ground state exists:

$$m^2 \gg R \text{ and } m^2 \gg H \longrightarrow \langle : \phi^2 : \rangle_\omega = \frac{1}{32\pi^2} m^2 + \beta R.$$

A semiclassical effect - III

$$\dot{H} = \frac{-H^4 + H_+^2 H^2}{H^2 - \frac{H_+^2}{4}}, \quad H_+^2 = 360\pi - 2880\pi^2 m^2 \beta.$$



On Hadamard states

AQFT seems to be a helpful tool in Cosmology, but

- is the result stable if we consider another kind of matter field [1]?
- Are all our assumptions really fulfilled?

Particularly does an Hadamard state exist on a FRW spacetime?

[1] Yes, C.D., Thomas-Paul Hack, Nicola Pinamonti, arXiv:0904.0612 [math-ph]

Existence of an Hadamard state - I

Let us consider a FRW spacetime with[2]

$$ds^2 = a^2(\tau)[-d\tau^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)],$$

let us restrict the class of scale factors as:

$$a(\tau) = -\frac{1}{H\,\tau} + O\left(\tau^{-2}\right) \ ,$$

$$\frac{da(\tau)}{d\tau} = \frac{1}{H\,\tau^2} + O\left(\tau^{-3}\right) \ , \frac{d^2a(\tau)}{d\tau^2} = -\frac{2}{H\,\tau^3} + O\left(\tau^{-4}\right).$$

- they all posses a cosmological horizon $\Im^- \sim \mathbb{R} \times \mathbb{S}^2$ in the past,
- If $a(\tau) = -\frac{1}{H\tau}$ then $\tau = -e^{-Ht}$, hence cosmological de-Sitter spacetime.
- as $\tau \to -\infty$, the background **"tends to"** de Sitter, Hence we are dealing with an exponential acceleration in the proper time t. This is the the prerequisite of all inflationary models.

[2]C.D., Nicola Pinamonti, V. Moretti Comm. Math. Phys **285** (2009), 1129

Existence of an Hadamard state - II

Let us consider the usual real scalar field

$$P\Phi_f = 0, \qquad P = -\Box + \xi R + m^2 \text{ and } \xi R + m^2 > 0$$

with smooth compactly supported initial datum f on an open set.

- Each solution Φ_f is a smooth function on M, *i.e.*, $\Phi_f \in C^\infty(M)$,
- The set of solutions $S(M)$ of our equation is a symplectic space if endowed with the Cauchy-independent nondegenerate symplectic form:

$$\sigma(\Phi_f, \Phi_g) \doteq \int_\Sigma \left(\Phi_f \nabla_N \Phi_g - \Phi_g \nabla_N \Phi_f\right) d\mu_g^{(\Sigma)},$$

- each $\Phi_f \in S(M)$ can be extended to a unique smooth solution of the same equation on $M \cup \Im^- \longrightarrow \Gamma\Phi_f \doteq \Phi_f|_{\Im^-} \in C^\infty(\Im^-)$.

Existence of an Hadamard state - III

Proposition

For all $\Phi_f \in S(M)$ and $m^2 + \xi R > 0$, then

- $\Gamma\Phi_f \in \left(\mathcal{S}(\Im^-), \sigma_{\Im^-}\right)$, where

$$\mathcal{S}(\Im^-) = \left\{\psi \in \ C^\infty(\mathbb{R}\times\mathbb{S}^2) \mid \psi \in L^\infty, \partial_l\psi \in L^1, \widehat{\psi} \in L^1, k\widehat{\psi} \in L^\infty\right\},$$

$$\sigma_{\Im^-}(\psi_1,\psi_2) = \int\limits_{\mathbb{R}\times\mathbb{S}^2} \left(\psi_1\partial_l\psi_2 - \psi_2\partial_l\psi_1\right) d\mu.$$

- $\sigma_{\Im^-}(\Gamma\Phi_f, \Gamma\Phi_g) = H^2\sigma(\Phi_f, \Phi_g)$,

We can introduce a *distinguished state* whose 2-point function is

$$\omega(\psi,\psi') = \int\limits_{\mathbb{R}\times S^2} 2k\Theta(k)\overline{\widehat{\psi}(k,\theta,\varphi)}\widehat{\psi}'(k,\theta,\varphi)dkdS^2(\theta,\varphi),$$

The End[3]

Consequence

For all $\Phi_f \in S(M)$ and $m^2 + \xi R > 0$, then

- the projection Γ induces a pull-back of any boundary state in the bulk:

$$\omega_M(\Phi_f, \Phi_g) \doteq \omega(\Gamma\Phi_f, \Gamma\Phi_g).$$

Main Result

The two-point function $\omega_M(f, g)$ arising from the distinguished state on $\Im^-$

- is always of Hadamard form,
- is the Bunch-Davies state in de Sitter spacetime,
- ω_M represents a natural distinguished cosmological "ground (vacuum) state" to tackle the study of linear perturbations,
- it is invariant under the natural action of any bulk isometry.

[3] C.D., Nicola Pinamonti, V. Moretti J. Math. Phys. **50** (2009) 062304, 0812.4033 [gr-qc]

Conclusions

We have realised

- AQFT helps to clear the haze present in Cosmology
- we can prove
  1. the existence of late time stable solutions for the semiclassical Einstein's equations,
  2. the identification of a distinguished ground state for free scalar field theories on certain FRW spacetimes.

And now,

- we can look for thermal and Hadamard states for all kinds of free field[4],
- we can look at inflation from a mathematical-physics point of view as a tool to rule out a few dozen models,
- we can try to understand the role quantum effects of all kind of fields when used to explain dark matter, dark energy,...

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[4] C.D., Thomas-Paul Hack and Nicola Pinamonti: work in progress