Distinguished ground states in FRW spacetimes

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Outline of the Talk

- Motivations, *i.e.*, trivia about cosmological models,
- On the geometry of the background and on the cosmological horizon,
- On the underlying field theory: form the bulk to the horizon,
- Constructing distinguished states,
- On the Hadamard property of these distinguished states.

Based on

- C. Dappiaggi, NP, V. Moretti CMP 2009
- C. Dappiaggi, NP, V. Moretti 0812.4033 [gr-qc]

Motivations - Part I (The Cosmos)

- The description of the Universe is based on the **Cosmological Principle**
- It yields that topologically: the background $M \sim I \times \Sigma$,
  - I is an open interval of $\mathbb{R}$ (*"cosmological time"*),
  - Σ are homegeneous $3D$ manifolds, topologically either a sphere, or a plane or a paraboloid.
- The geometry is given by a Friedmann-Robertson-Walker type of metric

$$g = -dt^2 + a^2(t)\left[\frac{dr^2}{1-\kappa r^2} + r^2 d\mathbb{S}^2(\theta,\varphi)\right]. \quad \kappa = 0, \pm 1$$

- Recent observations suggest (no definitive statement!)
  - $\kappa = 0$,
  - the cosmological constant is slightly negative, *i.e.*, $a(t) = e^{Ht}$.

Motivations - Part II (Early Universe)

- The standard Cosmological model is plagued by many problems such as
  1. homogeneity problem (ad hoc initial conditions),
  2. flatness problem,
  3. many other (model dependent) problems.
- Possible way out: assume a **phase of rapid expansion** during the early stages.
- Hence during these stages, the geometry is a FRW spacetime with $\kappa = 0$ and

  $$a(t) = e^{H_0 t} \qquad H_0 >> H_{now}$$

  We require also $N = \ln \frac{a(t_f)}{a(t_i)} \geq 60$.
- What drives the rapid expansion? A scalar field whose dynamic is driven by a suitable self-interacting potential

  This is inflation! At least 100 different models.

Motivations - Part III (What is left behind)

Independently from the chosen model, we end up with

- a reasonably cheap explanation of the problems of the Cosmological model,
- no hint on the origin of the scalar field from a fundamental theory.

Nonetheless the model has an advantage

- it also predicts effects observables nowadays: it contributes to anisotropies of the CMB.

Here comes the problem: How it affects the CMB?

To cut a long story short: the power spectrum of the quantum fluctuations of the scalar field are scale-free?

Our aim: to make such assertion precise, to understand the mathematical background we need to make inflation. a mathematically sound theory

First Step: Quantum fluctuations are calculated with respect to a ground state.

Does it exist?!

A distinguished class of "cosmological spacetimes" - I

Hyp. 1) Cosmological Principle $\Longrightarrow$

$$g_{FRW} = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 dS^2(\theta,\varphi)\right], \quad M \sim I \times X_3$$

and $a(t) \in C^\infty(I, R^+)$.

Immediate consequences:

1. Consider a co-moving observer as the integral line $\gamma(t)$ of ∂_t. If $M \setminus J^+(\gamma) \neq \emptyset$, then causal signal departing from each $x \in M \setminus J^+(\gamma)$ never reach $\gamma(t)$. Then we call $\partial J^+(\gamma)$ the (past) **cosmological horizon**,
2. if one introduces the conformal time $d\tau = \frac{dt}{a(t)}$ and rescales the metric as

$$g_{FRW} = a^2(\tau)\left[-d\tau^2 + \frac{dr^2}{1-\kappa r^2} + r^2 dS^2(\theta,\varphi)\right],$$

then τ ranges in $(\alpha, \beta) \subset \mathbb{R}$. Sufficient condition for the existence of an horizon is $\alpha > -\infty$ and/or $\beta < \infty$.

A distinguished class of "cosmological spacetimes" - I

Hyp. 2) We set $\kappa = 0$, *i.e.*, $M \equiv I \times \mathbb{R}^3$ hence the spacetime is conformally (a piece of) Minkowski.

Hyp. 3) We restrict the class of scale factors as:

$$a(\tau) = -\frac{1}{H\,\tau} + O\left(\tau^{-2}\right)\ ,$$

$$\frac{da(\tau)}{d\tau} = \frac{1}{H\,\tau^2} + O\left(\tau^{-3}\right)\ , \frac{d^2a(\tau)}{d\tau^2} = -\frac{2}{H\,\tau^3} + O\left(\tau^{-4}\right).$$

- Here H is chosen as *positive* and the interval $I \doteq (-\infty, 0)$.

Consequences and Properties - I

1. If $a(\tau) = -\frac{1}{H\tau}$ then $\tau = -e^{-Ht}$, hence
$$ds^2 = -dt^2 + e^{-2Ht}(dr^2 + r^2 d\mathbb{S}^2(\theta, \varphi)), \quad t \in (-\infty, \infty).$$
This is the cosmological de-Sitter spacetime.
2. For our choice of $a(\tau)$, as $\tau \to -\infty$, the background **"tends to"** de Sitter. Hence we are dealing with an exponential acceleration in the proper time t. This is the the prerequisite of all inflationary models.

Consequences and Properties - II

- There is always a **Cosmological horizon**. Under the coordinate change

$$U = \tan^{-1}(\tau - r)\ , \qquad V = \tan^{-1}(\tau + r),$$

the metric becomes:

$$g_{FRW} = \frac{a^2(U,V)}{\cos^2 U \cos^2 V}\left[-dUdV + \frac{\sin^2(U-V)}{4} dS^2(\theta,\varphi)\right].$$

Theorem:

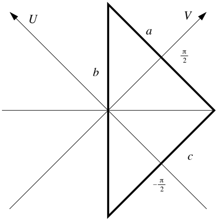
Under the previous assumptions the spacetime (M, g_{FRW}) can be extended to a larger spacetime $(\widehat{M}, \widehat{g})$ which is a conformal completion of the asymptotically flat spacetime at past (or future) null infinity $(M, a^{-2}g_{FRW})$, *i.e.*, "a" plays the role of the conformal factor. The cosmological horizon is

$$\Im^- \doteq \partial J^+(M; \widehat{M}) = \partial M.$$

Consequences and Properties - III

- Conformall null infinity $\Im^-$ corresponds to the horizon *(region c in the figure)* and it is a null degenerate manifold with

$$g|_{\Im^-} = 0 \cdot dl^2 + H^{-2}\left(d\mathbb{S}^2(\theta,\varphi)\right),$$



Furthermore the manifold $M \cup \Im^-$ enjoys:

1. the vector field ∂_τ is a conformal Killing vector for $\widehat{g}$ in M,
2. the vector ∂_τ becomes tangent to $\Im^\pm$ approaching it and coincides with $-H^{-1}\widehat{\nabla}^b a$.

Aim of the analysis:

We want to model a scalar QFT on a cosmological spacetime and we want to find a distinguished ground state

Hence from now on, we consider $\Phi : M \to \mathbb{R}$ such that

$$P\Phi = 0, \qquad P = -\Box + \xi R + m^2 \text{ and } \xi R + m^2 > 0$$

with compactly supported initial data on a Cauchy surface,

N.B. FRW spacetimes are globally hyperbolic, hence Cauchy problems are meaningful.

- Each solution Φ is a smooth function on M, *i.e.*, $\Phi \in C^\infty(M)$
- The set of solutions $S(M)$ of our equation is a symplectic space if endowed with the Cauchy-independent nondegenerate symplectic form:

$$\sigma(\Phi_1, \Phi_2) \doteq \int_\Sigma (\Phi_1 \nabla_N \Phi_2 - \Phi_2 \nabla_N \Phi_1)\, d\mu_g^{(\Sigma)}$$

More on classical solutions

Next Problem :

We want to better characterize the space of solutions $S(M)$

Any $\Phi \in \mathcal{S}(M)$ can be decomposed in modes ($\mathbf{k} \in \mathbb{R}^3$, $k = |\mathbf{k}|$,)

$$\Phi(\tau, \vec{x}) = \int_{\mathbb{R}^3} d^3\mathbf{k} \left[\phi_{\mathbf{k}}(\tau, \vec{x}) \widetilde{\Phi}(\mathbf{k}) + \overline{\phi_{\mathbf{k}}(\tau, \vec{x}) \widetilde{\Phi}(\mathbf{k})} \right],$$

with respect to the functions

$$\phi_{\mathbf{k}}(\tau, \vec{x}) = \frac{1}{a(\tau)} \frac{e^{i\mathbf{k}\cdot\vec{x}}}{(2\pi)^{\frac{3}{2}}} \chi_{\mathbf{k}}(\tau),$$

$\chi_{\mathbf{k}}(\tau)$, is solution of the differential equation

$$\frac{d^2}{d\tau^2}\chi_{\mathbf{k}} + (V_0(\mathbf{k}, \tau) + V(\tau))\chi_{\mathbf{k}} = 0,$$

$$V_0(\mathbf{k}, \tau) := k^2 + \left(\frac{1}{H\tau}\right)^2 \left[m^2 + 2H^2\left(\xi - \frac{1}{6}\right)\right], \quad V(\tau) = O(1/\tau^3).$$

- Furthermore it holds the normalization

$$\frac{d\overline{\chi_{\mathbf{k}}(\tau)}}{d\tau}\chi_{\mathbf{k}}(\tau)-\overline{\chi_{\mathbf{k}}(\tau)}\frac{d\chi_{\mathbf{k}}(\tau)}{d\tau}=i.\quad \forall\tau\in(-\infty,0)$$

Idea: Construct a general solution treating $V(\tau)$ as a perturbation potential over solutions with $V=0$, *i.e.* those in purely de-Sitter background.

Thus for $V(\tau)=0$

$$\chi^0_{\mathbf{k}}(\tau)=\frac{\sqrt{-\pi\tau}}{2}e^{\frac{i\pi\nu}{2}}\overline{H^{(2)}_{\nu}(-k\tau)},$$

with

$$\nu=\sqrt{\frac{9}{4}-\left(\frac{m^2}{H^2}+12\xi\right)},$$

where $H^{(2)}_{\nu}$ is the Hankel function of second kind.

Perturbative solutions in the general case

- Let us consider the retarded fundamental solutions $S_{\mathbf{k}}$ of

$$\frac{d^2}{d\tau^2}\chi^0_k(\tau) + (V_0(k,\tau))\chi^0_k(\tau) = 0$$

- Then the general solutions $\chi_{\mathbf{k}}$ can be constructed

$$\chi_{\mathbf{k}}(\tau) = \chi^0_{\mathbf{k}}(\tau) +$$

$$+\sum_{n=1}^{+\infty}(-1)^n \int_{-\infty}^{\tau} dt_1 \int_{-\infty}^{t_1} dt_2 \cdots \int_{-\infty}^{t_{n-1}} dt_n S_{\mathbf{k}}(\tau, t_1) S_{\mathbf{k}}(t_1, t_2)\cdots$$

$$S_{\mathbf{k}}(t_{n-1}, t_n) V(t_1) V(t_2)\cdots V(t_n)\chi_{\mathbf{k}}(t_n).$$

The series is convergent

if $|Re\nu| < 1/2$ and $V = O(\tau^{-3})$,
if $\frac{1}{2} \leq |Re\nu| < 3/2$ and $V = O(\tau^{-5})$.

From the bulk to the horizon ...

Bulk) A Weyl C^*-algebra $\mathcal{W}(M)$ can be associated to $(S(M),\sigma)$. This is, up to $*$-isomorphisms, unique and its non vanishing generators $W_M(\phi)$ satisfy:

$$W_M(-\phi) = W_M(\phi)^*, \quad W_M(\phi)W_M(\phi') = e^{\frac{i}{2}\sigma(\phi,\phi')}W_M(\phi+\phi'),$$

Horizon) The symplectic space of real wavefunctions is:

$$\mathcal{S}(\Im^-) = \left\{\psi \in C^\infty(\mathbb{R}\times\mathbb{S}^2) \mid \psi \in L^\infty, \partial_\ell\psi \in L^1, \widehat{\psi} \in L^1, k\widehat{\psi} \in L^\infty\right\},$$

$$\sigma_{\Im^-}(\psi,\psi') = \int\limits_{\mathbb{R}\times\mathbb{S}^2} \left(\psi\frac{\partial\psi'}{\partial\ell} - \psi'\frac{\partial\psi}{\partial\ell}\right). \quad \forall\psi,\psi' \in \mathcal{S}(\Im^-)$$

Algebra) Since $\sigma_{\Im^-}$ is nondegenerate, we can construct a Weyl C^*-algebra $\mathcal{W}(\Im^-)$ as

$$W_{\Im^-}(\psi) = W^*_{\Im^-}(-\psi), \qquad W_{\Im^-}(\psi)W_{\Im^-}(\psi') = e^{\frac{i}{2}\sigma_{\Im^-}(\psi,\psi')}W_{\Im^-}(\psi+\psi').$$

Distinguished state on $\Im^-$

We can introduce a *distinguished state* $\lambda : \mathcal{W}(\Im^-) \to \mathbb{C}$ unambiguously defined

$$\lambda\left(W(\psi)\right) = e^{-\frac{\mu(\psi,\psi)}{2}}, \quad \forall W(\psi) \in \mathcal{W}(\Im^-)$$

where $\forall \psi, \psi' \in \mathcal{S}(\Im^-)$

$$\mu(\psi,\psi') = \int\limits_{\mathbb{R}\times S^2} 2k\Theta(k)\overline{\widehat{\psi}(k,\theta,\varphi)}\widehat{\psi}'(k,\theta,\varphi)dkdS^2(\theta,\varphi),$$

being $\widehat{\psi}(k), \widehat{\psi}'(k)$ the Fourier-Plancherel transform

$$\widehat{\psi}(k) = \int\limits_{\mathbb{R}} dl\, \frac{e^{ikl}}{\sqrt{2\pi}}\psi(l,\theta,\varphi).$$

The state λ enjoys the following (almost straightforward) properties:

- it is quasifree and pure,
- referring to its GNS triple $(\mathcal{H}, \Pi, \Upsilon)$ it is invariant under the left action α of the horizon symmetry group.

Properties of λ

Let us consider the timelike future directed vector field ∂_τ whose projection on the horizon is $\widetilde{Y} \propto \partial_l$ (also a generator of the algebra of horizon simmetries). Then

- then λ is the **unique** quasifree pure state on $\mathcal{W}(\Im^-)$ which is invariant under $\alpha_{\exp(s\partial_l)}$ ($s \in \mathbb{R}$) and the unitary group implementing such representation leaving fixed the cyclic GNS vector is strongly continuous with nonnegative self-adjoint generator,
- Each folium of states on $\mathcal{W}(\Im^-)$ contains at most one pure state which is invariant under $\alpha_{\exp(s\partial_l)}$,
- and much more...

Back to the bulk

Notice: $\phi \in S(M)$ can be extended to a unique smooth solution of the same equation on $M \cup \Im^- \longrightarrow \Gamma\phi \doteq \phi|_{\Im^-} \in C^\infty(\Im^-)$.

Theorem 1

If $\phi \in S(M)$ and $0 < \epsilon < \frac{3}{2} - \nu$, then

- $\Gamma\phi$ decays faster than $1/l^\epsilon$ whereas $\partial_l \Gamma\phi$ faster than $1/l^{1+\epsilon}$,
- $\Gamma\phi \in \mathcal{S}(\Im^-)$,
- $\sigma_{\Im^-}(\Gamma\phi, \Gamma\phi') = H^2 \sigma(\phi, \phi')$.

Particularly it exists an isometric $*$-homomorphism:

$$\imath : \mathcal{W}(M) \to \mathcal{W}(\Im^-),$$
$$\imath\left(W(\phi)\right) \doteq W(\Gamma\phi).$$

Back to the bulk II

- Any state $\widetilde{\lambda} : \mathcal{W}(\Im^-) \to \mathbb{C}$ can be pulled back to $\imath^*(\widetilde{\lambda}) : \mathcal{W}(M) \to \mathbb{C}$.
- Particularly the preferred state

$$\lambda_M(a) := \lambda(\imath(a)). \quad \forall a \in \mathcal{W}(M)$$

- In the de Sitter spacetime, λ_M is the Bunch-Davies state,
- λ_M is considered by cosmologist as the "ground (vacuum) states" in the study of linear perturbations,
- it is invariant under the natural action of any bulk isometry Y on the algebra. The one-parameter U^Y_s group implementing such an action leaves fixed the cyclic vector in the GNS representation of λ_M,
- if Y is everywhere timelike and future-directed in M, then the 1-parameter group U^Y_s has positive self-adjoint operator.

Glimpses of Hadamard(ology)

Recall) A quasi-free state ω is fully characterized by its two-point function.

Local description: A two-point function $\omega(x, y)$ of a state ω is **Hadamard** if, for any normal neighbourhood $\mathcal{O}_p$,

$$\omega(x, y) = \frac{U(x, y)}{\sigma_\epsilon(x, y)} + V(x, y) \ln \sigma_\epsilon(x, y) + W(x, y).$$

Global description: using **microlocal analysis**, a state ω of a real smooth K.-G. field is of Hadamard form if and only if the Schwartz kernel of the two-point function satisfies

$$WF(\omega) = \left\{((x, k_x), (y, -k_y)) \in (T^*M)^2 \setminus 0 \mid (x, k_x) \sim (y, k_y), k_x \triangleright 0\right\}.$$

Is λ_M Hadamard?

• Hadamard states are distinguished since they mimic ultraviolet behaviour of the ground state in Minkowski!

To investigate λ_M, we first write its Schwarz kernel as the quadratic form

$$\lambda_M(f,g) = \int\limits_{\mathbb{R}\times\mathbb{S}^2} 2k\Theta(k)\overline{\widehat{\psi}_f(k,\theta,\varphi)}\widehat{\psi}_g(k,\theta,\varphi)dkd\mathbb{S}^2(\theta,\varphi),$$

where $\psi_f = \Gamma E(f)$ and $\psi_g = \Gamma E(g)$.

Theorem

λ_M inviduates a distribution on $\mathcal{D}'(M\times M)$ such that

$$WF(\lambda_M) = \mathcal{V} =$$

$$= \left\{((x,k_x),(y,-k_y)) \in (T^*M)^2 \setminus 0 \mid (x,k_x) \sim (y,k_y), k_x \triangleright 0\right\},$$

thus it is Hadamard.

On the inclusion $\supset$

Since it holds

$$\lambda_M(f \otimes Pg) = \lambda_M(Pf \otimes g) = 0, \qquad \lambda_M(f \otimes g) - \lambda_M(g \otimes f) = E(f \otimes g),$$

then the inclusion $\supset$ descends from $\subset$ by means of the theorem of propagation of sigularities proved by Hörmander (see Radzikowski and many others).

Sketch of the proof. $\subset$

- Let us read λ_M as follows: introduce

$$K = (T \otimes I)(\Gamma E \otimes \Gamma E) \in \mathcal{D}'((\Im^- \times \Im^-) \times (M \times M)),$$
$$T = \frac{1}{H^2\pi^2(l - l' - i\epsilon)^2} \otimes \delta(\theta - \theta')\delta(\varphi - \varphi').$$

- introduce a sequence of cut-off functions $\chi_n \in C_0^\infty(\Im^-;\mathbb{C})$ and

$$\lambda_n \doteq \mathcal{K}(\chi_n \otimes \chi_n),$$

where $\mathcal{K} : C_0^\infty(\Im^- \times \Im^-) \to \mathcal{D}'(M \times M)$ is the map associated with the kernel ${}^tK$ in view of Schwarz kernel theorem.

Big final fat Theorem:

The sequence λ_n are such that:

1. $WF(\lambda_n) \subset \mathcal{V}$
2. $\lambda_n \to \lambda_M$ in the weak sense in $\mathcal{D}'(M \times M)$
3. $\sup_n \sup_{k \in V} |k|^N |\widehat{h\lambda_n}| < \infty$ for all $N \geq 1$ and for all $h \in C_0^\infty(M \times M; \mathbb{C})$ where V is any cone closed in $(T^*M)^2 \setminus 0$ lying in the complement of $\mathcal{V}$.

Hence λ_M satisfies $\subset$ and its of Hadamard form.

What lies in front of us?

Summary:

- A distinguished Hadamard state for a scalar field theory exists in a large class of FRW backgrounds,
- These backgrounds are of cosmological relevance.
- It has interesting properties.

Open Questions:

- How can we connect this results to present observations?
- Is a free scalar field theory enough?[1]
- How can we describe interacting theories in our scenario?
- Is the road to mathematically precise inflationary models open?

[1] C.D., Klaus Fredenhagen and Nicola Pinamonti, Phys. Rev. D**77** (2008) 104015