A rigorous semiclassical effect in Cosmology

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Durban, 27th of May 2009

Outline of the Talk

- Introduction: Cosmology and (algebraic) quantum field theory,
- The set-up: From a massive scalar field to the conformal anomaly,
- Applications: On semiclassical Einstein's equations in cosmology,
- Robustness: On the conformal anomaly for Dirac fields.

Motivations - What we know

The 20^{th} century thought us a few good lessons:

1) Interactions $\longrightarrow$ quantum field theory on **flat spacetime**:

- it works almost perfectly for free and electroweak forces,
- perturbative QFT, renormalization, etc...

2) Gravitational interaction $\longrightarrow$ **General Relativity**.

3) Algebraic approach, $\longrightarrow$ also allows for a rigorous discussion of QFT on curved backgrounds [Brunetti, Dimock, Fredenhagen, Hollands, Kay, Verch, Wald,...]

Natural playground $\longrightarrow$ Cosmology

- unveils the structure and dynamics of the Universe,
- we can fully use QFT on curved background in the algebraic approach.

The Cosmological Principle and FRW - I

We wish to model the geometry of the Universe!

Occam's razor leads to the Cosmological principle, *i.e.*,

- spacetime is homogeneous $\longrightarrow$ at each instant of time, all space points look the same,
- spacetime is isotropic $\longrightarrow$ there is at each point an observer who sees an isotropic Universe.

This entails

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1+kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right].$$

- the parameter $k = 0, \pm 1$ tells me if spatial section are flat planes, spheres or hyperbolas,
- there is still no dynamical content. This determines $a(t)$ and, to this avail, one needs a good $T_{\mu\nu}$, to solve

$$R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}.$$

The Cosmological Principle and FRW - II

Which $T_{\mu\nu}$? Let us start with *classical matter*

We know:

- part of mass-energy in the Universe is ordinary matter (stars, galaxies, clusters),
- their density is so low that they appear like "dust" with density ρ. Hence

$$T_{\mu\nu} = \rho\zeta_\mu\zeta_\nu \quad \zeta^\mu\zeta_\mu = 1$$

- if we also include a contribution from pressure, then

$$T_{\mu\nu} = \rho\zeta_\mu\zeta_\nu + P\left(g_{\mu\nu} + \zeta_\mu\zeta_\nu\right),$$

which is the stress-energy tensor of a **perfect fluid**. This is the the most general choice for $T_{\mu\nu}$ if the matter is classical.

Dynamics of the scale factor

Dynamics is encoded in Einstein's equations

$$R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}.$$

I assume from now on $\Lambda = 0$!

$$G_{tt} = R_{tt} - \frac{R}{2} g_{tt} = 8\pi T_{tt} \longrightarrow 3\frac{\dot{a}^2 + k}{a^2} = 8\pi\rho,$$

$$G_{xx} = R_{xx} - \frac{R}{2} g_{xx} = 8\pi T_{xx} \longrightarrow 3\frac{\ddot{a}}{a^2} = -4\pi(\rho + 3P).$$

Conservation of $T_{\mu\nu}$, *i.e.*, $\nabla^\mu T_{\mu\nu} = 0$ yields

$$\dot{\rho} + 3(\rho + P)\frac{\dot{a}}{a} = 0.$$

Notice that the dynamical content boils down to this last equation and to an identity between traces:

$$Tr(G_{\mu\nu}) = -R = 8\pi Tr(T_{\mu\nu}).$$

Dynamics of the scale factor - II

To solve that system we need an **equation of state** $\rho = \gamma P$

We assume from now on $k = 0$, but only for simplicity of the talk!

Eq. of state	scale factor	conservation of $T_{\mu\nu}$
Dust, $P = 0$	$a(t) \propto t^{\frac{2}{3}}$	$\rho a^3(t) = const.$
Radiation, $P = \frac{\rho}{3}$	$a(t) \propto \sqrt{t}$	$\rho a^4(t) = const.$

One should interpret the results, but instead let us look at the assumptions.

To get here we assumed

1. isotropy and homogeneity of the Universe,
2. the behaviour of the stress-energy tensor is classical,
3. pressure and energy density are related by an equation of state.

Are we happy?

Practical and Foundational Problems

The "classical" approach to cosmology is highly unsatisfactory

on a practical ground,

- the model is far too rough in the description of matter,
- it is plagued by many problems, namely
    1. the singularity problem as $a \to 0$, namely $\rho \to \infty$,
    2. the flatness problem,
    3. the homogeneity problem.

and a foundational ground,

- Classical matter cannot account for an explanation of interac. as QFT.

What we would like to know

The quest to solve those problems prompted

- Modern Cosmology $\longrightarrow$ matter is often modelled by a scalar field.
- Bright side
  1. it takes seriously the role of QFT as the natural playground to discuss (quantum) matter and interactions,
  2. it provides a nice exit to most of the problems of standard cosmology,
  3. models of inflation lead to testable consequences, on the temperature of CMB in particular.
- Dark side
  1. still plenty of open problems (dark matter, dark energy...),
  2. it is unclear how to derive these models from "first principles",
  3. unclear concepts in curved backgrounds:(temperature, termal equilibrium)...

The main question

Can we explain any of the unclear phenomena just with a rigorous analysis?

Do we really need new models or, in some cases, what we have is enough?

Let us investigate **dark energy**[1] $\longrightarrow$ semiclassical Einstein's equations!

[1] C.D., Klaus Fredenhagen, Nicola Pinamonti, Phys. Rev. D**77** (2008) 104015

The framework

Let us look at our ingredients:

- We fix the background as a FRW spacetime with flat spatial section

$$ds^2 = -dt^2 + a^2(t)[dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)], \quad M \equiv \mathbb{R} \times \mathbb{R}^3$$

- we consider for "simplicity of the talk" a scalar field on M

$$P\phi(x) \doteq \left(\Box_g - \frac{R}{6} - m^2\right)\phi(x) = 0,$$

which is conformally coupled to scalar curvature.

- we shall seek solutions of $G_{\mu\nu} = 8\pi\langle : T_{\mu\nu} :\rangle_\omega$, $\longrightarrow$ in FRW

$$-R = 8\pi\langle : T :\rangle_\omega .$$

Intermezzo: the quest for an Hadamard state

What is a good choice for ω?

A physically reasonable choice is

1. an ω which is quasi-free (technical condition),
2. an ω which is of Hadamard form,
   - same ultraviolet behaviour as the ground state in Minkowski,
   - only on these states the quantum fluctuations of $T_{\mu\nu}$ are finite.

Hence in a geodesic normal neighbourhood of any point $p \in \mathbb{R} \times \mathbb{R}^3$, the integral kernel of the two-point function is

$$\omega(x,y) = \frac{U(x,y)}{\sigma_\epsilon(x,y)} + V(x,y) \ln \frac{\sigma_\epsilon(x,y)}{\lambda} + W(x,y).$$

Intermezzo - II

Let start again from

$$\omega(x,y) = \frac{U(x,y)}{\sigma_\epsilon(x,y)} + V(x,y)\ln\frac{\sigma_\epsilon(x,y)}{\lambda} + W(x,y).$$

One can prove that

- U, V, W are all smooth scalar functions,
- in Minkowski $U = 1$ and $V = \frac{m^2}{2\sqrt{m^2(x-y)^2}} J_1(\sqrt{m^2(x-y)^2})$, whereas in curved backgrounds they are a series

$$U(x,y) = \sum_{n=0}^{\infty} u_n(x,y)\sigma^n, \quad V(x,y) = \sum_{n=0}^{\infty} v_n(x,y)\sigma^n,$$

  determined out of recursion relations,
- the singular part, namely U and V, depends only on geometric quantities such as $R, R^2, R_{\mu\nu}R^{\mu\nu}\ldots$
- the choice of a quantum state of Hadamard form lies only in W.

On the stress-energy tensor

Let us *assume* there exists an Hadamard state!

The classical stress-energy tensor can be written as

$$T_{\mu\nu} = \nabla_\mu\phi(x)\nabla_\nu\phi(x) - \frac{1}{2}g_{\mu\nu}\nabla_\rho\phi(x)\nabla^\rho\phi(x) + \left(\frac{1}{6}G_{\mu\nu} + g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\right)\phi^2(x).$$

- It is **conserved** $\longrightarrow \nabla^\mu T_{\mu\nu} = 0$,
- for $m = 0$ and $\zeta = \frac{1}{6}$ it is **traceless** $T = 0$.

Conservation is also required for consistency with Einstein's equations since

$$\nabla^\mu G_{\mu\nu} = 0.$$

Does it hold also at a quantum level?

The quantum stress-energy tensor - I

Semiclassical Einstein's equations are

$$G_{\mu\nu} = 8\pi\langle : T_{\mu\nu} : \rangle_\omega ,$$

hence the left hand side is still conserved. This implies

$\nabla^\mu \langle : T_{\mu\nu} : \rangle_\omega$, but is it true?

- The answer is no! The expectation value becomes proportional to terms as

$$\langle : \phi P \phi : \rangle_\omega \propto [v]_1 \doteq v_1(x,x), \quad \langle : (\nabla^\mu \phi) P \phi : \rangle_\omega \propto \nabla^\mu [v]_1 ,$$

A way out: exploit a freedom in the definition of $T_{\mu\nu}$.

The quantum stress-energy tensor - II

Let us introduce the modified stress-energy tensor

$$T^{\eta}_{\mu\nu}(x) = T_{\mu\nu}(x) + \eta g_{\mu\nu}(x)\phi(x)P\phi(x) \quad \eta \in \mathbb{R},$$

where $P = \Box_g - \frac{R}{6} - m^2$.

- the new term is dynamically sterile, *i.e.*, it vanishes on shell.
- the new term is conserved and traceless on shell,
- the new term gives rise to a non vanishing contribution at a quantum level,

$$\nabla^{\mu}\langle : \phi P \phi : \rangle_{\omega} \neq 0.$$

Theorem:[2] If ω is an Hadamard state, then, if and only if $\eta = \frac{1}{3}$, it holds

$$\nabla^{\mu}\langle : T_{\mu\nu} : \rangle_{\omega} = 0.$$

[2] V. Moretti: Commun. Math. Phys. **232** (2003) 189

The trace anomaly

In order to have a conserved $T_{\mu\nu}$, there is a price to pay!

$$\langle :T: \rangle_\omega = -m^2 \frac{W(x,x)}{8\pi^2} + \frac{v_1(x,x)}{4\pi^2},$$
$$v_1(x,x) = \frac{1}{720}(R_{ij}R^{ij} - \frac{R^2}{3} + \Box R) + \frac{m^4}{8}.$$

This is the so-called conformal anomaly!

- it contains a term proportional to $\Box R \longrightarrow$ dynamically unstable.

We can use a remaining freedom: add to $T_{\mu\nu}$ conserved tensors $t_{\mu\nu}$ built out of curvature invariants:

$$t_{\mu\nu} = \frac{\delta}{\delta g_{\mu\nu}} \int_M d^4x \sqrt{|g|}(CR^2 + DR_{\mu\nu}R^{\mu\nu}).$$

- Their trace $t^\mu_\mu \propto \Box R$ (it recalls $f(R)$ theories).

A semiclassical effect

Let us plug the trace anomaly in

$$-R = 8\pi\langle : T : \rangle_\omega .$$

Let us shake the equations a little bit and we end up with

$$-6\left(\dot{H} + 2H^2\right) = -8\pi m^2 \langle : \phi^2 : \rangle_\omega + \frac{1}{\pi}\left(-\frac{1}{30}(\dot{H}H^2 + H^4 + \frac{m^4}{4}\right),$$

where $H = \frac{\dot{a}(t)}{a(t)}$.

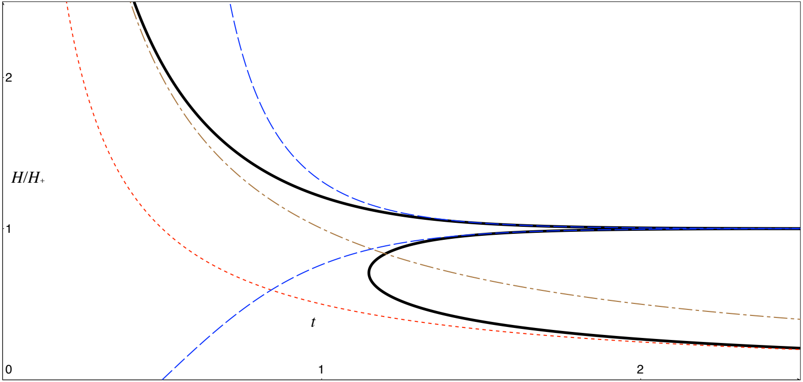
A notion of approximate ground state exists (WKB approximation):

$$m^2 \gg R \text{ and } m^2 \gg H^2 \longrightarrow \langle : \phi^2 : \rangle_\omega = \frac{1}{32\pi^2} m^2 + \beta R. \quad \beta \in \mathbb{R}$$

This yields

$$\dot{H} = \frac{-H^4 + H_+^2 H^2}{H^2 - \frac{H_+^2}{4}}, \quad H_+^2 = 360\pi - 2880\pi^2 m^2 \beta.$$

A semiclassical effect - II



Is the result robust enough?

One should doubt what I told you!

1. Does an Hadamard state really exist on a FRW spacetime?
2. What happens with other kind of fields?

Let us look at this second problem! Notice that

- Fermionic fields might behave in a really different way,

- the result is ultimately tied to the existence and to the form of the conformal anomaly.

Our goal: calculate the anomaly for Dirac fields[3].

[3] C. D., T. P. Hack and N. Pinamonti, arXiv:0904.0612 [math-ph].

Basic Geometric Structures

We shall work here in any 4D globally hyperbolic spacetime!

The following structures are necessary and well-defined

- the spin group $Spin(3,1)$ as

$$\{e\} \longrightarrow \mathbb{Z}_2 \longrightarrow Spin(3,1) \longrightarrow SO(3,1) \longrightarrow \{e\},$$

- the frame bundle, over M endowed with a non degenerate metric

$$S(M) \doteq S(M)[Spin(3,1), \tilde{\pi}, M] \quad \tilde{\pi} : S(M) \to M,$$

with a bundle hom. $\rho : S(M) \to F(M)[SO(3,1), \pi', M]$,

- the Dirac bundle is an associated bundle

$$DM \doteq S(M) \times_T \mathbb{C}^4, \quad D^*M \doteq S(M) \times_T (\mathbb{C}^4)^*$$

out of the repr. $T \doteq D^{(\frac{1}{2},0)} \oplus D^{(0,\frac{1}{2})}$ of $Spin_0(3,1) \sim SL(2,\mathbb{C})$ on $\mathbb{C}^4$.

Classical Dynamic - I

Kinematical configurations: A **Dirac spinor** and a **cospinor** is

$$\psi \in \Gamma(DM), \quad \psi^\dagger \in \Gamma(D^*M).$$

• All space and time oriented $4D$ globally hyperbolic spacetimes admit a spin structure

The Dirac (and the dual) bundle trivializes and hence

$$\psi : M \longrightarrow \mathbb{C}^4 \quad \psi^\dagger : M \longrightarrow \mathbb{C}^4.$$

• Next ingredient are γ-matrices, the building block of the algebra of $Spin(3,1)$,

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu},$$

Classical Dynamic - II

- Natural covariant derivative ∇ on DM as a pull-back from that on $T(M)$,

$$\nabla : \Gamma(DM) \to \Gamma(DM \otimes T^*(M)).$$

- It is remarkable $\nabla\gamma = 0$,
- we call **dynamically allowed** a (co)spinor such that

$$(-\gamma^\mu \nabla_\mu + m)\,\psi = 0, \quad (\gamma^\mu \nabla_\mu + m)\,\psi^\dagger = 0,$$

  - $D = -\gamma^\mu \nabla_\mu + m$ is the **Dirac operator**,
  - $D' = \gamma^\mu \nabla_\mu + m$ is the **dual Dirac operator**.

- since $DD' = D'D = -\Box + \frac{R}{4} + m^2$ then

$$\begin{cases} D\psi = 0, \longrightarrow (-\Box + \frac{R}{4} + m^2)\psi = 0 \\ D'\psi^\dagger = 0 \longrightarrow (-\Box + \frac{R}{4} + m^2)\psi^\dagger = 0 \end{cases}.$$

From Classical to Quantum Stress-Energy Tensor

The classical stress-energy tensor is

$$T_{\mu\nu} = \frac{1}{2}\left(\psi^\dagger \gamma_{(\mu}\psi_{;\nu)} - \psi^\dagger_{(;\mu}\gamma_{\nu)}\psi\right) - \frac{1}{2}L[\psi]g_{\mu\nu}.$$

- Dirac equation entails

$$\nabla^\mu T_{\mu\nu} = 0 \quad T = g^{\mu\nu}T_{\mu\nu} = -m\psi^\dagger\psi.$$

- We are interested in $\langle : T_{\mu\nu} : \rangle_\omega$ with an Hadamard state ω.
  - point-splitting along a geodesic

$$T_{\mu\nu}(x,y) \doteq \frac{1}{2}\left(\psi^\dagger(x)\gamma_{(\mu}g^{\nu'}_{\nu)}\psi(y)_{;\nu'} - \psi^\dagger(x)_{;(\mu}\gamma_{\nu)}\psi(y)\right),$$

Seeking a quantum conserved $T_{\mu\nu}$

- Subtraction of the singularity

$$\omega(:T_{\mu\nu}(x):) \doteq Tr\left[\omega(T_{\mu\nu}(x,y)) - T^{sing}_{\mu\nu}(x,y)\right]_{y=x}$$

$$\doteq Tr\left[D^0_{\mu\nu}\left(\omega^-(x,y) + \frac{1}{8\pi^2}D'_y H\right)\right]_{y=x} \doteq \frac{1}{8\pi^2} Tr\left[D_{\mu\nu}W(x,y)\right]_{y=x}$$

- Canonical but unsatisfactory choice of $D^0_{\mu\nu}$, $D_{\mu\nu}$

$$D^0_{\mu\nu} \doteq \frac{1}{2}\gamma_{(\mu}\left(g^{\nu'}_{\nu)}\nabla_{\nu'} - \nabla_{\nu)}\right), \qquad D_{\mu\nu} \doteq -D^0_{\mu\nu}D'_y,$$

- Problem: $D'_y H(x,y)$ does not satisfy Einstein's equations
  - $\langle :T_{\mu\nu}:\rangle_\omega$ is not conserved $\rightarrow$ ill-posed semiclassical Einstein's equations
  - we can seek for the same solution as in the scalar case
    1. we add to the classical $T_{\mu\nu}$ multiples of $L[\psi]g_{\mu\nu}$,
    2. it amounts to $D^{mod}_{\mu\nu} = D_{\mu\nu} + \frac{c}{2}g_{\mu\nu}(D'_x + D_y)D'_y$.

The Trace

Let us

- consider the described modified $T_{\mu\nu}^{mod}$,
- an Hadamard state ω,
- a reference length $\lambda = 2\exp(\frac{7}{2} - 2\gamma)m^{-2}$ if $m \neq 0$ (arbitrary for $m = 0$),

It turns out that if $c = \frac{1}{6}$

- $\nabla^\mu \langle : T_{\mu\nu} : \rangle_\omega = 0$,
- the trace does not vanish[4] even with $m = 0$ and

$$\langle : T : \rangle_\omega = \frac{1}{\pi^2}\left(\frac{R^2}{1152} - \frac{\Box R}{480} - \frac{R_{\mu\nu}R^{\mu\nu}}{720} - \frac{7}{5760}R_{\mu\nu\rho\delta}R^{\mu\nu\rho\delta}\right).$$

• We have the same structure of the conformal anomaly as for the scalar case!

[4] We admit abuses of Ph.D. students in the realization of this project.

Conclusions

Hurdled Problems

- it is possible to tackle cosmological problems with the language of AQFT,
- this approach shows
  1. the existence of late time stable solutions for the semiclassical Einstein's equations,
  2. clarification of the origin of the conformal anomaly for Dirac fields,
  3. the identification of a distinguished ground state for free scalar field theories on certain FRW spacetimes (next talk).

Problems yet to hurdle

- identify a good notion of termal equilibrium at least in FRW spacetimes (using N. Pinamonti, 0806.0803 to appear on CMP),
- create a valuable companion tool of experiment with the aim to rule out the pathological cosmological models,
- understand the role of quantum effects for all kind of fields in phenomena such as dark matter, dark energy, baryogenesis ...