A novel point of view on the conformal anomaly for quantised Dirac fields[1]

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[1]C. D., Thomas Hack and Nicola Pinamonti, to appear soon

Motivations - I

We have already heard[2]

- if we consider a scalar field ϕ

$$\left(\Box_g - \frac{R}{6} - m^2\right)\phi = 0,$$

- on a FRW spacetime

$$ds^2 = -dt^2 + a^2(t)\left[dr^2 + r^2 d\mathbb{S}^2(\theta,\varphi)\right],$$

- we can dwell into a semiclassical analysis

$$G_{\mu\nu} = 8\pi\langle : T_{\mu\nu} :\rangle_\omega, \quad \longrightarrow \quad -R = 8\pi\langle : T :\rangle_\omega$$

[2] Please refer to Nicola Pinamonti's talk

Motivations - II

Classically we know that

$$T = -m^2\phi^2(x),$$

but, at a quantum level, life is hard, and on a FRW spacetime

$$\langle : T : \rangle_\omega = -m^2 \langle : \phi^2 : \rangle_\omega + \frac{1}{720}\left(R_{\mu\nu}R^{\mu\nu} - \frac{R^2}{3} + \Box R \right) + \frac{m^4}{8}.$$

The semiclassical Einstein's equation with $H = \frac{\dot{a}(t)}{a(t)}$

$$-6(\dot{H}^2 + 2H^2) = -m^2 \langle : \phi^2 : \rangle_\omega + \frac{1}{30}(-\frac{1}{\pi}(\dot{H}H^2 + H^4 + \frac{m^4}{4})).$$

For $m^2 \gg H$ and $m^2 \gg R \longrightarrow \langle : \phi^2 : \rangle_\omega \sim \frac{m^2}{32\pi^2} + \beta R$

$$\dot{H} = \frac{-H^4 + H_+^2 H^2}{H^2 - \frac{H_+^2}{4}} \quad H_+^2 = 360\pi - 2880\pi^2 m^2 \beta.$$

Starting Whistle

- the trace anomaly leads to an effective cosmological constant,
- it can be interpreted as a potential dark energy,
- de Sitter appears as a late time stable solution.

Question: Is this result stable for other kind of matter, spinors in particular?

Basic Geometric Structures

We shall work in a 4D globally hyperbolic spacetime.

The following structures are necessary and well-defined

- the spin group $Spin(3,1)$ as

$$\{e\} \longrightarrow \mathbb{Z}_2 \longrightarrow Spin(3,1) \longrightarrow SO(3,1) \longrightarrow \{e\}\,,$$

- the frame bundle, over M endowed with a non degenerate metric

$$F(M) \doteq F(M)[SO(3,1), \pi', M] \quad \pi' : F(M) \to M$$

- the spin structure over M is the pair $(S(M), \rho)$

$$S(M) \doteq S(M)[Spin(3,1), \tilde{\pi}, M] \quad \tilde{\pi} : S(M) \to M,$$

  with a bundle hom. $\rho : S(M) \to F(M)$,

- the Dirac bundle is an associated bundle

$$DM \doteq S(M) \times_T \mathbb{C}^4, \quad D^*M \doteq S(M) \times_T (\mathbb{C}^4)^*$$

  out of the repr. $T \doteq D^{(\frac{1}{2},0)} \oplus D^{(0,\frac{1}{2})}$ of $Spin_0(3,1) \sim SL(2,\mathbb{C})$ on $\mathbb{C}^4$.

Classical Dynamic - I

Kinematical configurations: A **Dirac spinor** and a **cospinor** is

$$\psi \in \Gamma(DM), \quad \psi^\dagger \in \Gamma(D^*M)$$

• All globally space and time oriented $4D$ globally hyperbolic spacetimes admit a spin structure

The Dirac (and the dual) bundle trivializes and hence

$$\psi : M \longrightarrow \mathbb{C}^4 \quad \psi^\dagger : M \longrightarrow \mathbb{C}^4.$$

• Next ingredient are γ-matrices, the building block of the algebra of $Spin(3,1)$,

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu},$$

Classical Dynamic - II

- Natural covariant derivative ∇ on DM as a pull-back from that on $T(M)$,

$$\nabla : \Gamma(DM) \to \Gamma(DM \otimes T^*(M)).$$

- It is remarkable $\nabla\gamma = 0$
- we call **dynamically allowed** a (co)spinor such that

$$(-\gamma^\mu \nabla_\mu + m)\,\psi = 0, \quad (\gamma^\mu \nabla_\mu + m)\,\psi^\dagger = 0,$$

  - $D = -\gamma^\mu \nabla_\mu + m$ is the **Dirac operator**,
  - $D' = \gamma^\mu \nabla_\mu + m$ is the **dual Dirac operator**.

- since $DD' = D'D = -\Box + \frac{R}{4} + m^2$ then

$$\begin{cases} D\psi = 0, \longrightarrow (-\Box + \frac{R}{4} + m^2)\psi = 0 \\ D'\psi^\dagger = 0 \longrightarrow (-\Box + \frac{R}{4} + m^2)\psi^\dagger = 0 \end{cases}$$

Fundamental Solutions

- $P = -\Box + \frac{R}{4} + m^2$ is an hyp. operator with metric principal symbol,

This entails:

- P admits an advanced E^+ and retarded E^- fundamental solution,
- $S^\pm \doteq D' E^\pm$ are the advanced and retarded fundamental solutions of D

$$S^\pm : \Gamma_0(DM) \to \Gamma(DM),$$
$$supp(S^\pm f) \subset J^\pm(supp(f)) \quad \forall f \in \Gamma_0(DM).$$

- with identical properties we have $S^\pm_* \doteq DE^\pm$ for D',
- we call causal propagator

$$S \doteq S^+ - S^- \quad S_* \doteq S^+_* - S^-_*.$$

Field Algebra - I

Ingredients for the field algebra:

- doubling $\longrightarrow \widetilde{\Gamma}_0 \doteq \Gamma_0(DM) \oplus \Gamma_0(D^*M)$,
- Conjugation $\mathcal{C}(f \oplus f') = f^\dagger \oplus f'^\dagger$ for all $f \oplus f' \in \widetilde{\Gamma}_0$,
- global pairing of $\Gamma(DM)$ (and $\Gamma(D * M)$) as

$$\langle \psi, \psi' \rangle \doteq \int_M \psi(x) \left(\psi'\right)(x) d\mu(x),$$

- positive sesquilinear product on $\widetilde{\Gamma}_0 / Ker(\widetilde{S})$ with $\widetilde{S} = S \oplus S_*$,

$$\left\{\widetilde{f}, \widetilde{h}\right\}_{\widetilde{S}} = -i\langle f_1^\dagger, Sh_1\rangle + i\langle S_* f_2, h_2^\dagger\rangle,$$

  for all $\widetilde{f} = f_1 \oplus f_2$ and $\widetilde{h} = h_1 \oplus h_2$ in $\widetilde{\Gamma}_0$,
- Hilbert space as the completion $\mathcal{H} \doteq \overline{\widetilde{\Gamma}_0 / Ker(\widetilde{S})}$.

Field Algebra II

The unital $*$-algebra of fields is $\mathcal{F}(M, g)$

1. elements are $B(\widetilde{f})$ where $\widetilde{f} \to B(\widetilde{f})$ is linear $\forall \widetilde{f} \in \widetilde{\Gamma}_0$,
2. $B(Df \oplus D'f') = 0$, for all $\widetilde{f} \doteq f \oplus f' \in \widetilde{\Gamma}_0$,
3. $B(\mathcal{C}\widetilde{f}) = B(\widetilde{f})^*$,
4. $B(\widetilde{f})^* B(\widetilde{h}) + B(\widetilde{h}) B(\widetilde{f})^* = \left\{\widetilde{f}, \widetilde{h}\right\}_{\widetilde{S}}$ (CAR).

We want observables to commute if supported at spacelike separated regions.

Algebra of observable $\mathcal{A}(M, g)$ is the even subalgebra of $\mathcal{F}(M, g)$.

Hadamard states - I

We seek a gauge invariant state $\omega : \mathcal{A}(M,g) \to \mathbb{C}$ such that

- it is positive $\to \omega(a^* a) \geq 0$ for all $a \in \mathcal{A}(M,g)$ and $\omega(\mathbb{I}) = 1$,
- it is quasifree

$$\omega^+(f,g) \doteq \omega(\psi(g)\psi^\dagger(f)) \text{ and } \omega^-(f,g) \doteq \omega(\psi^\dagger(f),\psi(g)),$$

- it is of Hadamard form.

Hadamard states

- have the same UV structure of Minkowski vacuum,
- are such that fluctuations of $T_{\mu\nu}$ are bounded.

Hadamard States - II

Hadamard states can be characterized

- locally $\rightarrow$ in a geodesic convex neighbourhood

$$\omega^{\pm}(x,y) = \pm \frac{1}{8\pi^2} D'_y \left(H^{\pm}(x,y) + W(x,y) \right),$$
$$H^{\pm}(x,y) = \frac{U(x,y)}{\sigma_\epsilon(x,y)} + V(x,y) \ln \frac{\sigma_\epsilon(x,y)}{\lambda}.$$

  - $\sigma(x,y)$ is the squared geodesic distance and λ a reference (squared) length,
  - U and V are smooth functions

$$U(x,y) = \sum_{n=0}^{\infty} u_n(x,y)\sigma^n, \quad V(x,y) = \sum_{n=0}^{\infty} v_n(x,y)\sigma^n,$$

  The coefficients u_n and v_n are determined via recursion relations!

- We have all ingredients to define normal ordering and the algebra of Wick polynomials.

From Classical to Quantum Stress-Energy Tensor

The classical stress-energy tensor is

$$T_{\mu\nu} = \frac{1}{2}\left(\psi^\dagger \gamma_{(\mu}\psi_{;\nu)} - \psi^\dagger_{(;\mu}\gamma_{\nu)}\psi\right) - \frac{1}{2}L[\psi]g_{\mu\nu}, \quad (1)$$

$$L[\psi] = \frac{1}{2}\left[\psi^\dagger D\psi + (D'\psi^\dagger)\psi\right]. \quad (2)$$

- Dirac equation entails

$$\nabla^\mu T_{\mu\nu} = 0 \quad T = g^{\mu\nu}T_{\mu\nu} = -m\psi^\dagger\psi.$$

- We are interested in $\langle : T_{\mu\nu} : \rangle_\omega$ with an Hadamard state ω.
  - point-splitting along a geodesic

$$T_{\mu\nu}(x,y) \doteq \frac{1}{2}\left(\psi^\dagger(x)\gamma_{(\mu}g^{\nu'}_{\nu)}\psi(y)_{;\nu'} - \psi^\dagger(x)_{;(\mu}\gamma_{\nu)}\psi(y)\right),$$

Seeking a quantum conserved $T_{\mu\nu}$

- Subtraction of the singularity

$$\omega(:T_{\mu\nu}(x):) \doteq Tr\left[\omega(T_{\mu\nu}(x,y)) - T^{sing}_{\mu\nu}(x,y)\right]_{y=x}$$

$$\doteq Tr\left[D^0_{\mu\nu}\left(\omega^-(x,y) + \frac{1}{8\pi^2}D'_y H\right)\right]_{y=x} \doteq \frac{1}{8\pi^2} Tr\left[D_{\mu\nu}W(x,y)\right]_{y=x}$$

- Canonical but unsatisfactory choice of $D^0_{\mu\nu}$, $D_{\mu\nu}$

$$D^0_{\mu\nu} \doteq \frac{1}{2}\gamma_{(\mu}\left(g^{\nu'}_{\nu)}\nabla_{\nu'} - \nabla_{\nu)}\right), \qquad D_{\mu\nu} \doteq -D^0_{\mu\nu}D'_y,$$

- Problem: $D'_y H(x,y)$ does not satisfy Einstein's equations
  - $\langle : T_{\mu\nu} : \rangle_\omega$ is not conserved $\rightarrow$ ill-posed semiclassical Einstein's equations,
  - we can seek for the same solution as in the scalar case
    1. we add to the classical $T_{\mu\nu}$ multiples of $L[\psi]g_{\mu\nu}$,
    2. it amounts to $D^{mod}_{\mu\nu} = D_{\mu\nu} + \frac{c}{2}g_{\mu\nu}(D'_x + D_y)D'_y$.

The Trace

Let us

- consider the described modified $T^{mod}_{\mu\nu}$,
- an Hadamard state ω,
- a reference length $\lambda = 2\exp(\frac{7}{2} - 2\gamma)m^{-2}$ if $m \neq 0$ (arbitrary for $m = 0$),

It turns out that if $c = \frac{1}{6}$

- $\nabla^\mu \langle : T_{\mu\nu} : \rangle_\omega = 0$,
- the trace does not vanish even with $m = 0$ and

$$\langle : T : \rangle_\omega = \frac{1}{\pi^2}\left(\frac{R^2}{1152} - \frac{\Box R}{480} - \frac{R_{\mu\nu}R^{\mu\nu}}{720} - \frac{7}{5760}R_{\mu\nu\rho\delta}R^{\mu\nu\rho\delta}\right).$$

Conclusions

We have learned

- to control the classical and quantum aspects of Fermi fields,
- to rigorously calculate the trace anomaly,
- to construct the extended algebra encompassing Wick polynomials.

We shall

- extend the semiclassical cosmological analysis for the scalar field,
- use our knowledge to tackle problems in baryogenesis and leptogenesis,
- many many many other things...