From Spin Groups and Modular P_1CT Symmetry to Covariant Representations and the Spin-Statistics Theorem

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Abstract

Starting from the assumption of modular P_1CT symmetry in quantum field theory a representation of the universal covering of the Poincaré group is constructed in terms of pairs of modular conjugations. The modular conjugations are associated with field algebras of unbounded operators localised in wedge regions. It turns out that an essential step consists in characterising the universal covering group of the Lorentz group by pairs of wedge regions, in conjunction with an analysis of its geometrical properties.

In this thesis two approaches to this problem will be developed in four spacetime dimensions. First a realisation of the universal covering as the quotient space over the set of pairs of wedge regions will be presented. In spite of the intuitive definition, the necessary properties of a covering space are not straightforward to prove. But the geometrical properties are easy to handle. The second approach takes advantage of the well-known features of spin groups, given as subgroups of Clifford algebras. Characterising elements of spin groups by pairs of wedge regions is possible in an elegant manner. The geometrical analysis is performed by means of the results achieved in the first approach.

These geometrical properties allow for constructing a representation of the universal cover of the Lorentz group in terms of pairs of modular conjugations. For this representation the derivation of the spin-statistics theorem is straightforward, and a PCT operator can be defined. Furthermore, it is possible to transfer the results to nets of field algebras in algebraic quantum field theory with ease.

Many of the usual assumptions in quantum field theory like the spectrum condition or the existence of a covariant unitary representation, as well as the assumption on the quantum field to have only finitely many components, are not required. For the standard axioms, the crucial assumption of modular P_1CT symmetry constitutes no loss of generality because it is a consequence of these.

Zusammenfassung

Ausgehend von der Annahme der modularen P_1CT-Symmetrie in der Quantenfeldtheorie wird eine unitäre Darstellung der universellen Überlagerung der Poincarégruppe aus Paaren von modularen Konjugationen konstruiert. Die modularen Konjugationen sind zu Feldalgebren unbeschränkter Operatoren assoziiert, welche in Keilgebieten lokalisiert sind. Ein wesentlicher Schritt hierfür besteht in der Charakterisierung der universellen Überlagerung der Lorentzgruppe durch Paare von Keilgebieten, verbunden mit einer Analyse der geometrischen Eigenschaften dieser Charakterisierung.

Hierfür werden in dieser Arbeit im Fall von vier Raumzeit-Dimensionen zwei Möglichkeiten hergeleitet. Zunächst wird eine Realisierung der universellen Überlagerung als Quotientenraum über der Menge von Keilpaaren vorgestellt. Trotz der intuitiven Definition sind die notwendigen Eigenschaften einer universellen Überlagerung nicht in einfacher Weise zu zeigen. Dagegen ist die geometrische Struktur sehr gut zu handhaben. Ein zweiter Zugang beruht auf den bekannten Eigenschaften von Spingruppen als Untergruppen von Cliffordalgebren. Die Identifizierung mit Paaren von Keilregionen lässt sich direkt etablieren. Die geometrischen Eigenschaften werden mit Hilfe der Resultate für die zuerst erwähnte Konstruktion hergeleitet.

Die geometrischen Eigenschaften beider Realisierungen erlauben es, eine unitäre kovariante Darstellung der universellen Überlagerung mit Paaren von modularen Konjugationen zu kontruieren. Für diese Darstellung lässt sich das Spin-Statistik-Theorem auf direktem Wege beweisen. Ein PCT-Operator kann ebenfalls definiert werden. Des Weiteren ist es ohne großen Aufwand möglich, die Resultate auf Netze von Feldalgebren im Rahmen der algebraischen Quantenfeldtheorie zu übertragen.

Annahmen, wie die Existenz einer unitären Darstellung, welche die Spektrumsbedingung erfüllt, oder eine endliche Zahl von Komponenten des Quantenfeldes werden nicht benötigt. Die entscheidende Annahme der modularen P_1CT-Symmetrie bedeutet im Rahmen der Standardaxiome keine Einschränkung der Allgemeinheit, da sie aus diesen folgt.

Contents

1 Introduction

Two fundamental theorems in quantum field theory have been of recurrent interest to physicists from the very beginnings of the theory in the 1930s up to the present. The first is the spin-statistics theorem and the second is the PCT theorem. The spin-statistics theorem connects two different characteristics of particles, their spin and the property whether or not they obey the Pauli exclusion principle. The PCT theorem is the statement that under very weak assumptions any quantum field theory possesses a general symmetry: Invariance under simultaneous reflection of parity, charge and time. Both theorems and their proofs are closely tied together in their development. Advances in one of them often led to advances for the other, and there is a mutual interplay: Some authors were able to derive either of them from the other. One can say that the early results culminated in 1940 with the work of Pauli [Pau40] who derived the spin-statistics theorem for free quantum field theories obeying some basic principles like locality, Poincaré invariance and positivity of the energy.

The formulation of a set of axioms for quantum field theory by Gårding and Wightman [GW65] allowed for the structural analysis and the proof of rigorous results in a model-independent manner. This triggered new interest in the fundamental theorems of quantum field theory. In this setting Burgoyne [Bur58] and Lüders and Zumino [LZ58] achieved a proof of the spin-statistics theorem and of the PCT theorem from general physical assumptions.

The Wightman axioms emanated from the analysis of the n-point functions of quantum field theories [GW65, Wig57a, Wig57b]. In their usual formulation [SW00] an assumption on the number of components of the quantum field has to be made; only fields with finitely many components are allowed. Counterexamples of Streater [Str67] and Oksak and Todorov [OT68] show that simply dropping this assumption admits models violating the spin-statistics theorem or the PCT theorem. It is natural to ask whether it is possible to generalise the setting in such a way that the essential theorems and structural results are preserved also for fields with infinitely many components, since there does not seem to exist any general physical argument to exclude these.

In [Kuc05], a theorem of Bisognano and Wichmann was used to generalise the Wightman framework accordingly. Bisognano and Wichmann proved two properties of Wightman quantum field theories that may seem surprising at first sight [BW75, BW76]. These properties have been termed modular P_1CT symmetry and modular covariance because of their relations to the modular theory of Tomita and Takesaki

[Tak02]. They are formulated as follows. For a Wightman quantum field F the algebra of field operators located in the spacetime region $\mathcal{O}$ will be denoted by $\mathbf{F}(\mathcal{O})$.

Modular $\mathbf{P_1CT}$ symmetry. The modular conjugation J_W associated with the field algebra $\mathbf{F}(W)$ of the wedge W acts geometrically as the product of a charge conjugation, reflection in time and reflection in one spatial direction.[1] This operation is called P_1CT conjugation and can be expressed as

$$J_W \mathbf{F}(\mathcal{O}) J_W = \mathbf{F}^t(j_W \mathcal{O}).$$

j_W is the reflection in the edge of the wedge W and $\mathcal{O}$ an open spacetime region. The superscript t denotes a twist on the field due to the statistics operator.

Modular covariance. The modular group associated with the field algebra of a wedge W implements boosts on the field algebras $\mathbf{F}(\mathcal{O})$ located in open spacetime regions $\mathcal{O}$ in the form

$$\Delta_W^{it} \mathbf{F}(\mathcal{O}) \Delta_W^{-it} = \mathbf{F}(\Lambda_W(2\pi t)\mathcal{O}),$$

where Δ_W is the modular operator associated with the wedge algebra $\mathbf{F}(W)$, and $\Lambda_W(t)$ is conjugated to the boost $\Lambda_{W_R}(t)$ in the x_1-direction with rapidity t.[2]

Modular covariance is not only the essence of the Bisognano-Wichmann theorem, but also has a clear-cut physical explanation in the Unruh effect [Unr76] connected with the KMS-condition. This has been observed by Sewell [Sew80, Sew82]. Unruh investigated the motion of an accelerated observer in Minkowski spacetime and showed that this observer measures the vacuum state as a thermal heat bath. On the other hand a thermal equilibrium state is characterised by the KMS-condition, and the dynamics of a KMS-state coincides with the modular group [HHW67]. So assuming modular covariance amounts to assuming the physically well-understood Unruh effect to hold.

In [Kuc05] the assumption of a finite number of components and even the assumptions of covariance and of the spectrum condition were dropped, and only modular P_1CT symmetry, being a consequence of the original axioms, was assumed. For this generalisation of the Wightman axioms a unitary representation of the universal covering of the rotation group has been constructed from pairs of modular conjugations. This representation satisfies the spin-statistics theorem and also entails

[1] A wedge is any Poincaré transform of the right wedge $W_R := \{x \in \mathbb{R}^{1+3};\ x_1 > |x_0|\}$ in Minkowski space (see figure 2.1 on page 9). The edge of the wedge W_R is the spacelike plane orthogonal to x_0 and x_1.

[2] This will be defined in eq. (2.2).

the existence of a PCT operator. An important step in the proof was a suitable characterisation of the universal covering of the rotation group in terms of pairs of wedges and reflections.

The main result accomplished in the present thesis is the extension of the program carried out in [Kuc05] to the much more involved cases of Lorentz invariant and Poincaré invariant quantum field theories. Starting point is the generalisation of the Wightman setup presented in Section 2.3.3. Neither a finite number of components of the quantum field nor the spectrum condition and existence of a covariant representation of the double cover of the Lorentz group are assumed, but only modular $\mathrm{P_1CT}$ symmetry. As mentioned above, this symmetry under reflection in time and one space direction combined with charge conjugation is a consequence of the standard Wightman axioms as proven by Bisognano and Wichmann. The idea is to define a representation of the universal cover of the Lorentz group in terms of pairs of modular conjugations associated with wedges, motivated by the observation that the product of two such modular conjugations implements a Lorentz transformation of the field. This can be seen by applying modular $\mathrm{P_1CT}$ symmetry twice and checking that the product of two reflections in edges of wedges is a restricted Lorentz transformation. In [BDFS00, BS04] it has been shown that it is possible to construct a representation of the restricted Lorentz group for a "bosonic" net of algebras. Unitary operators $U(\Lambda) := J_{W_1} J_{W_2}$ have been defined for a pair of reflections j_{W_1}, j_{W_2} generating the Lorentz transformation $\Lambda = j_{W_1} j_{W_2}$. Two main problems have to be solved. First, the independence of $J_{W_1} J_{W_2}$ of the representing pair of wedges has to be verified, or an adequate choice of reflections has to be given for every Λ. Second, one has to prove that the product of four modular conjugations can be reduced to a product of two modular conjugations in such a way that U is a homomorphism. This was successfully carried out in [BDFS00, BS04]. For the case of the universal cover $\tilde{L}_+^\uparrow$ of the Lorentz group in four spacetime dimensions, which is of primary interest here, a similar strategy is applied. In the bosonic case for two wedges with a common edge, which therefore generate the same reflection, the modular conjugations coincide. In general, this does not hold any more.

The approach taken in this thesis is to characterise $\tilde{L}_+^\uparrow$ by pairs of wedges rather than by pairs of reflections in edges of wedges. This naturally leads to a realisation of $\tilde{L}_+^\uparrow$ by equivalence classes of pairs of wedges. The definition of such a model, which will be called $\mathbf{G}_L$, is straightforward. The difficulty lies in proving that $\mathbf{G}_L$ is indeed the universal covering group of $\tilde{L}_+^\uparrow$. This is established by analysing first the geometrical properties of $\mathbf{G}_L$ and then the topological ones. A second possibility to describe $\tilde{L}_+^\uparrow$ by pairs of wedges will be given, which in some respect is more elegant because it is tied to the well-known description of spin groups as subgroups of Clifford algebras associated with orthogonal groups. Spin groups are two-sheeted covering groups, and the covering map associates reflections in hyperplanes to vectors. The identification of the spin groups with pairs of wedges is based on this property. It turns out that both approaches are closely related and the work invested in $\mathbf{G}_L$ pays off in the analysis of the spin groups.

In either of these characterisations any pair of wedges determines an element of the universal covering group of $L_+^\uparrow$. Both can be used to define a covariant unitary representation $\tilde{W}$ of $\tilde{L}_+^\uparrow$ by setting $\tilde{W}(g) = J_{W_1} J_{W_2}$ for two wedges specifying $g \in \tilde{L}_+^\uparrow$. This is done by adapting the proofs given in [BS04] for the bosonic case. It is worth mentioning that the product $J_{W_1} J_{W_2}$ is independent of the choice of wedges W_1, W_2 in the equivalence class characterising g. Employing the representation of the translation group defined in [BS04], the representation of $\tilde{L}_+^\uparrow$ can be extended to the universal cover of the restricted Poincaré group $\mathcal{P}_+^\uparrow$. This representation is shown to satisfy the spin-statistics theorem by the same method as in [Kuc05].

The assumption of modular $\mathrm{P_1CT}$ symmetry, valid in the original Wightman framework, is also inspired by results in algebraic quantum field theory. Algebraic quantum field theory is an axiomatic formulation which was put forward by Haag, Kastler and Araki (see [Haa96] and references therein). The focus in this approach is on the observables located in spacetime regions and algebras of bounded operators generated by them. Unobservable quantities like fermionic fields only enter at a later stage if representations of the abstract algebras are analysed. Consequently, the Bose-Fermi alternative is not required for the net of observables, and a priori no assumption on the number of components of a field is made.

Since this setting perfectly fits the modular theory of Tomita and Takesaki [Tak02], one would expect that the Bisognano-Wichmann properties continue to hold. For two-dimensional theories obeying wedge duality [Bor92, Flo98] and conformal theories [BGL93] this has been established. A general result for four-dimensional Poincaré-invariant theories is not at hand. But it has been proven that, if the modular objects act geometrically at all in a very general sense, then they act in the expected way suggested by the Bisognano-Wichmann theorem [Kuc97, Kuc01]. The converse approach has proven to be very successful: Starting from the assumption of modular $\mathrm{P_1CT}$ symmetry [Kuc95, Kuc98b] or modular covariance [GL95] it is possible to prove an algebraic spin-statistics theorem and PCT theorem.

On the on hand, Brunetti, Guido and Longo assumed modular covariance and constructed a unitary representation of the double covering of the Poincaré group with positive energy [BGL95, GL95] under which a field algebra with normal commutation relations transforms covariantly. This representation satisfies the spin-statistics theorem, and a PCT operator can be defined. Additionally the relation between modular $\mathrm{P_1CT}$ symmetry and modular covariance was clarified because Guido and Longo [GL95] also showed that modular covariance implies modular $\mathrm{P_1CT}$ symmetry.

Kuckert [Kuc95, Kuc98b] on the other hand assumed modular $\mathrm{P_1CT}$ symmetry and a compact group of internal symmetries. This implies uniqueness of the covariant representation of the Poincaré group. The spin-statistics theorem then holds for the field algebra.

Either of the approaches has its advantages. The assumption of modular P_1CT symmetry is weaker[3] than assuming modular covariance, but has to be supplemented by the assumption of a compact gauge group. The results presented in [KL07] and in this thesis improve upon both, because the assumption of modular P_1CT symmetry suffices to define a representation of the universal covering of the Poincaré group in terms of modular conjugations for which the spin-statistics theorem holds.

Taking into account the counterexamples of Streater [Str67] to the spin-statistics theorem and Oksak and Todorov [OT68] to the PCT theorem for infinite component fields, the need for an additional assumption like modular P_1CT symmetry in algebraic quantum field theory does not come as a surprise. These counterexamples arise because of a "wrong" choice of the representation of the Poincaré group. The construction of a representation in terms of modular unitaries Δ^{it} or modular conjugations J selects a canonical representation for which the spin-statistics theorem and the PCT theorem hold.

The generalisation of the Wightman framework proposed here, besides being powerful enough to yield a general spin-statistics theorem and the PCT theorem, is also flexible enough to transfer the results to the case of a field algebra associated with a net of algebras of observables in algebraic quantum field theory. This is possible if normal commutation relations hold in the field algebra. Compared to the work of Guido and Longo, who achieved the same result, the approach taken here is more general, because the assumption of modular P_1CT symmetry is weaker than the assumption of modular covariance, as remarked above. But the results are restricted to four spacetime dimensions up to now. The analysis of the Poincaré group in other spacetime dimensions, or even of other symmetry groups in different spacetimes, may yield further insights.

The physical and mathematical framework for this thesis is presented in Chapter 2, starting with the Wightman axioms in Section 2.1. The relevant structural results in this setting are the PCT theorem (Section 2.1.1), the Bisognano-Wichmann theorem (Section 2.1.2) and the spin-statistics theorem. The latter is discussed in Section 2.3 in the Wightman formalism and in the framework of algebraic quantum field theory. Modular theory and its role in (algebraic) quantum field theory is the general topic of Section 2.2. The algebraic approach to quantum field theory is introduced in Section 2.2.2. A brief outline of spin groups which are needed in the definition of the representations of the universal covering groups of the rotation and Lorentz group is given in Section 2.4.

Chapter 3 contains a discussion of the results presented in [Kuc05] for the universal covering of the rotation group and the derivation of the spin-statistics theorem in this setting. It serves as a preparation for the more involved case of Lorentz or Poincaré invariance and illustrates the general ideas. What is new compared to

[3] See [BDFS00, Chapter 5.3] for an example of a net satisfying modular P_1CT symmetry but not modular covariance.

[Kuc05] is the use of spin groups, simplifying some of the proofs.

The central results achieved in this thesis are contained in Chapter 4. There the universal covering group of the Lorentz group is constructed by introducing an appropriate equivalence relation on pairs of wedge regions. Spin groups are closely related to reflections in hyperplanes. It will be shown that they can, by a suitable identification, also be characterised by pairs of wedges. This is discussed in Sections 3.2.2 and 4.2.3. Both characterisations are appropriate to construct a unitary representation of the double cover of the Lorentz group which acts covariantly on the field (Section 4.3). Then, by the same method of proof as in the case of rotations, the spin-statistics theorem follows. The work in Sections 4.1, 4.2.1 and 4.3 is based on a collaboration with B. Kuckert and has been published in [KL07]. The simplifying tool of spin groups and their identification with pairs of wedges to define the representation is the topic of [Lor07].

The extension to the universal cover of the restricted Poincaré group defined in Chapter 5 is the final step towards the desired result and has not been published before. Besides the spin-statistics theorem, in this chapter the existence of a PCT operator is proven in a similar fashion as in [Kuc05].

2 Quantum Field Theory, Spin and Statistics, and Modular Theory

2.1 Quantum Field Theory and the Wightman Axioms

In 1925, Born and Jordan examined the problem of computing the energy radiated by an atom which changes its state. This led to the development of a quantum theory of the electromagnetic field. The quantisation of classical field theories was accompanied by heuristic arguments and considerations and usually the question of consistency stays unanswered. The growing interest in establishing model-independent results on firm mathematical grounds led to the development of *axiomatic quantum field theory* in the in the 1950's. The goal is to find a set of axioms expressing physical properties that any quantum field theory is expected to have in mathematical terms. Starting from a set of axioms allows one to analyse structural properties of quantum field theories independent of concrete models, specific equations of motion or particular interactions. Several approaches emerged, of which the *Wightman axioms* and *algebraic quantum field theory* (AQFT) based on ideas of Haag and Kastler are two prominent examples. Both approaches allow for proving fundamental theorems like the PCT theorem and the spin-statistics theorem to be introduced later. The great advantage of these axiomatic approaches to quantum field theory is accompanied by a severe drawback. Up to now there are only a few interacting models known which are given non-perturbatively. These are defined in two or three spacetime dimensions (see for example [GJ72, GJ81, BK04, Lec06a, Lec06b]), whereas an example of an interacting theory in four spacetime dimensions is still missing.

After having fixed the notation, a short introduction to the Wightman axioms will be given. For a brief outline of the principles underlying algebraic quantum field theory see Section 2.2.2.

We will try to formulate as much as possible in a coordinate-independent manner. Four-dimensional Minkowski space $\mathbb{R}^{1+3}$ is a four-dimensional real vector space equipped with a symmetric, nondegenerate bilinear form g, the metric, of signature $(+,-,-,-)$. The vectors in Minkowski space can be classified as being timelike ($g(x,x) > 0$), spacelike ($g(x,x) < 0$), lightlike ($g(x,x) = 0, x \neq 0$) or null ($x = 0$). A vector e with $g(e,e) = \pm 1$ is called a unit vector, and x is orthogonal to y if

$g(x,y) = 0$. A set of four orthogonal unit vectors is a basis of $\mathbb{R}^{1+3}$ and contains one timelike and three spacelike vectors. The standard basis on $\mathbb{R}^{1+3}$ consists of the vectors

$$e_0 = (1,0,0,0), \quad e_1 = (0,1,0,0), \quad e_2 = (0,0,1,0), \quad e_3 = (0,0,0,1).$$

For two vectors x, y, the expansion in terms of the standard basis with coefficients $(x^\mu)_{\mu=0\ldots3}$, $(y^\nu)_{\nu=0\ldots3}$ leads to the usual expression $g(x,y) = x^0y^0 - x^1y^1 - x^2y^2 - x^3y^3$ for the inner product.

A timelike unit vector e_0 specifies a time direction in the following sense. Any vector x can be uniquely decomposed into a part parallel to e_0 and a part orthogonal to e_0 and we can introduce the future light cone as

$$V^+ := \{x \in \mathbb{R}^{1+3};\ g(x,x) > 0 \text{ and } g(x,e_0) > 0\}.$$

Any spacetime point in the future light cone can be reached by a signal sent by an observer located at the origin. No information can be exchanged between two spacelike separated points.

The *Lorentz group* L is the set of linear maps on $\mathbb{R}^{1+3}$ leaving the inner product invariant, i.e. $g(x,y) = g(\Lambda x, \Lambda y)$ for all $x, y \in \mathbb{R}^{1+3}$ and $\Lambda \in L$. It has four connected components of which we denote the connected component containing the identity, called *restricted Lorentz group*, by $L_+^\uparrow$. The other components are $L_+^\downarrow, L_-^\downarrow$ and $L_-^\uparrow$, the sign referring to a positive or negative determinant and the downward-pointing arrow indicating time reversal. Consequently, L_+ denotes the union of $L_+^\uparrow$ and $L_+^\downarrow$.

In the mathematical literature the Lorentz group is denoted by $O(\mathbb{R}^4, g)$ since it is the orthogonal group associated with the symmetric nondegenerate bilinear form g on the vector space $\mathbb{R}^4$. In this notation L_+ corresponds to $SO(\mathbb{R}^4, g)$ and the restricted Lorentz group $L_+^\uparrow$ is $SO(\mathbb{R}^4, g)_0$. The index denotes the connected component of the identity. For brevity, sometimes $L_+^\uparrow$ is also denoted by L_0. The restricted Lorentz group $L_+^\uparrow$ maps the forward light cone into itself. There are various possibilities to describe the universal cover of the Lorentz group of which one example is the group $SL(2,\mathbb{C})$. The covering map will be denoted by $\tilde{\lambda}$.

The Poincaré group $\mathcal{P}$ is the group of maps from $\mathbb{R}^{1+3}$ to $\mathbb{R}^{1+3}$ satisfying $g(x - y, x - y) = g(Px - Py, Px - Py)$. It is the semidirect product[1] of L with the translation group $T \cong \mathbb{R}^{1+3}$, so $\mathcal{P} = L \ltimes \mathbb{R}^{1+3}$. Akin to the Lorentz group one uses the notation $\mathcal{P}_+^\uparrow$ for the *restricted Poincaré group* $L_+^\uparrow \ltimes \mathbb{R}^{1+3}$.

In the following, by a Lorentz transformation we will always mean a restricted Lorentz transformation, and state exceptions explicitly. The familiar notions of *rotation* and *boost* are not coordinate-independent. To give sense to these objects, one has to specify a time direction by a timelike unit vector e_0. Then a rotation is a

[1]For the definition of a semidirect product see Chapter 5.1.

Lorentz transformation which leaves e_0 and one spacelike vector invariant. A boost leaves two spacelike vectors orthogonal to e_0 fixed. In a given reference frame any Lorentz transformation μ has a unique decomposition[2] into a rotation ρ and a boost β in the form

$$\mu = \rho\beta.$$

It is important to notice that this decomposition depends on the frame of reference specified through the timelike unit vector e_0. Furthermore, if ρ and β commute in one reference frame, the decomposition into ρ' and β' in another reference frame does not need to commute any more.

An important set of regions in Minkowski space is the set of *wedge regions* $\mathcal{W}$.

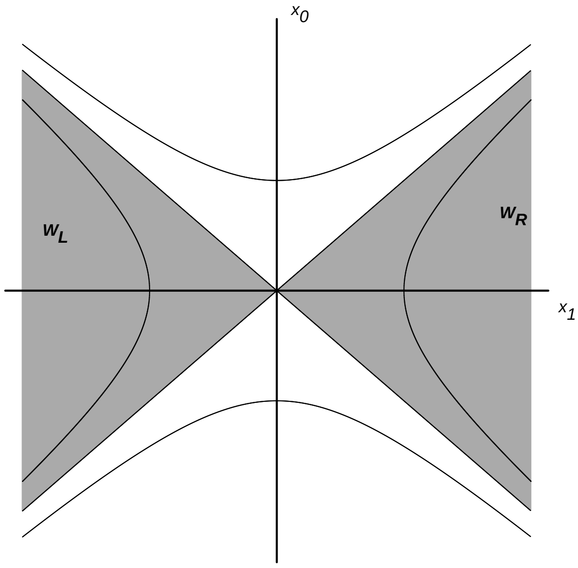


Figure 2.1: The right and left wedges W_R and W_L in the standard basis of Minkowski space. The projection of the positive and negative unit hyperboloid is indicated.

For example, in standard Minkowski space coordinates the right wedge W_R is given by

$$W_R := \{x \in \mathbb{R}^{1+3}, x_1 > |x_0|\}. \tag{2.1}$$

The left wedge $W_L := \{x \in \mathbb{R}^{1+3}, x_1 < -|x_0|\}$ is the set of points lying spacelike to all points in the closure of the right wedge. Denoting the set of points spacelike separated from another set (the *causal complement*) by a prime and the closure of a set by a bar, this can be expressed as $W_L = \overline{W_R}'$. Figure 2.1 shows the projection of the right and left wedges W_R and W_L onto the x_0, x_1 plane. The unit hyperboloid is

[2]This is related to the polar decomposition of closed operators on a Hilbert space into a unitary and a positive operator, see also page 63.

also indicated. The set of future directed elements of the timelike unit hyperboloid is denoted by M_1^+. An observer on this hyperboloid who is accelerated by the one-parameter group of boosts in the x_1 direction moves along the plotted line. This one-parameter group of boosts is given by

$$\Lambda_R(t) = \begin{pmatrix} \cosh(t) & \sinh(t) & 0 & 0 \\ \sinh(t) & \cosh(t) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.2}$$

The parameter t is referred to as the *rapidity* of the boost. Wedge regions, in the following referred to as *wedges*, can conveniently be described by specific pairs vectors in a coordinate-independent way. The set of *zweibeine* Z is

$$Z := \{\xi := (t_\xi, x_\xi) \in \mathbb{R}^{1+3} \times \mathbb{R}^{1+3};\ g(t_\xi, t_\xi) = 1, g(x_\xi, x_\xi) = -1, g(t_\xi, x_\xi) = 0\}. \tag{2.3}$$

So a zweibein is an ordered pair of a timelike and a spacelike unit vector. Any zweibein $\xi = (t_\xi, x_\xi)$ defines a wedge W_ξ via

$$W_\xi := \{x \in \mathbb{R}^{1+3}; -g(x, x_\xi) > |g(x, t_\xi)|\}, \tag{2.4}$$

and the set of all these wedges is denoted by $\mathcal{W}$. The Lorentz group acts transitively on the set of zweibeine and therefore it acts transitively on the set of wedges. This means that $\mathcal{W} = \{\Lambda W_\xi; \Lambda \in L_+^\uparrow\}$ for any $\xi \in Z$. One can easily check that

$$\Lambda W_\xi = W_{\Lambda\xi}.$$

The *edge of a wedge* W_ξ is the two-dimensional spacelike subspace of $\mathbb{R}^{1+3}$ orthogonal to t_ξ and x_ξ. The *stabiliser group* $\mathfrak{S}(W_\xi)$ of the wedge W_ξ is the group of Lorentz transformations leaving W_ξ invariant. The edge of the wedge is also invariant under the stabiliser group. The right wedge W_R has an abelian stabiliser group generated by rotations around the x_1 axis and boosts in the x_1 direction. This implies that the stabiliser group is abelian for any wedge, since all stabiliser groups of wedges are conjugated ($\mathfrak{S}(W_{\Lambda\xi}) = \Lambda\mathfrak{S}(W_\xi)\Lambda^{-1}$). Usually one introduces the set of wedges as the set of regions $\{PW_R, P \in \mathcal{P}_+^\uparrow\}$. We consider also certain subsets of these and introduce the following notation:

$$\mathcal{W}_{e_0} := \{W_\xi;\ \xi \in Z \text{ and } t_\xi = e_0\}, \tag{2.5}$$
$$\mathcal{W} := \{W_\xi;\ \xi \in Z\}, \tag{2.6}$$
$$\mathcal{W}_P := \{PW_\xi;\ \xi \in Z \text{ and } P \in \mathcal{P}_+^\uparrow\}. \tag{2.7}$$

$\mathcal{W}_{e_0}$ will be relevant in Chapter 3. The set $\mathcal{W}$ will be referred to as the set of wedges located at the origin. Only in Chapter 5 the full set of wedges $\mathcal{W}_P$ will be required.

The space of *test functions* $\mathcal{S}$ is chosen as the space of smooth functions from $\mathbb{R}^4$ (equipped with the usual Euclidean norm) to $\mathbb{C}$ which, together with their derivatives, decrease faster than any polynomial when going to infinity in any direction. This choice is not of importance, the set $C_0^\infty(\mathbb{R}^4)$ of smooth functions with compact support would be equally well-suited.

Now the Wightman axioms for a quantum field theory will be introduced. Since the primary objects of interest in the present thesis are quantum fields with infinitely many components, a generalised version is needed. A simple generalisation proposed by Streater [Str67] is given in Section 2.1.3. The refined version which was used in [Kuc05, KL07] and in this thesis and which is based on the assumption of modular $\mathrm{P}_1\mathrm{CT}$ symmetry is presented in Section 2.3.3.

The Wightman axioms are based on works of Wightman [Wig57a, Wig57b] and were formulated by Gårding and Wightman in 1952, but have been published only in [GW65].

An important reference is the monograph [SW00] by Streater and Wightman. In [Str75] one can find a discussion of possible modifications of the axiomatic setup and their consequences.

Axiom A: Pure physical states correspond to rays in a Hilbert space $\mathcal{H}$.

Axiom B: There exists a continuous unitary representation U of the universal cover of the Poincaré group $\tilde{\mathcal{P}}_+^\uparrow \cong SL(2,\mathbb{C}) \ltimes \mathbb{R}^{1+3}$.[3] The representation of the translation subgroup has four generators P^μ, $\mu = 0,\ldots,3$, which correspond to the energy-momentum. The square $P^\mu P_\mu = m^2$ corresponds to the mass of the particle, and the joint spectrum of the operators P^μ is contained in the closure of the forward light cone V^+. This condition is referred to as the *spectrum condition*.

Axiom C: The *vacuum vector* $\Omega \in \mathcal{H}$ is a distinguished state, unique up to a phase factor. It is invariant under U.

A second set of axioms deals with the notion of a field and its transformation law.

Axiom D:

1. A field transforming under an n-dimensional matrix representation D of $\tilde{\mathcal{P}}_+^\uparrow$ is a set of n operators $F_1,\ldots,F_n$ from the space $\mathcal{S}$ of test functions to (unbounded) operators in $\mathcal{H}$. For every $f \in \mathcal{S}$, the operators $F_1(f),\ldots,F_n(f)$ are densely defined. There exists a dense domain $\mathcal{D}$ in $\mathcal{H}$ which is contained in the domains of all $F_1(f),\ldots,F_n(f)$ and their adjoint operators $F_1(f)^\dagger,\ldots,F_n(f)^\dagger$. $\mathcal{D}$ is mapped into itself under all field operators, their adjoints and under all unitary operators $U(g)$, $g \in \tilde{\mathcal{P}}_+^\uparrow$. Furthermore, $\mathcal{D}$ contains the vacuum vector Ω.

[3] A definition of the semidirect product is given on page 83.

2. For every $i \in 1 \ldots, n$ and $\Phi, \Psi \in \mathcal{D}$, the map $f \mapsto \langle \Phi, F_i(f)\Psi \rangle$ from $\mathcal{S}$ to $\mathbb{R}$ is a tempered distribution.
3. The field F transforms covariantly under the representation U, i.e.

$$U(\{a, A\})F_i(f)U(\{a, A\})^\dagger = \sum_k D_{ik}(A^{-1})F_k(f_{\{a,A\}}),$$

where $f_{\{a,A\}}(x) := f(\tilde{\lambda}(A^{-1})(x-a))$. Recall that $\tilde{\lambda}$ was the covering map from $SL(2, \mathbb{C})$ to $L_+^\uparrow$.

The algebra generated by all operators $F_i(f)|_{\mathcal{D}}$ and $F_i(f)^\dagger|_{\mathcal{D}}$ with $i = 1, \ldots, n$ is denoted by $\mathbf{F}$ and referred to as the algebra of field operators. Accordingly the algebra of operators $\mathbf{F}(\mathcal{O})$ located in an open region $\mathcal{O}$ of Minkowski space is the $*$-subalgebra of $\mathbf{F}$ with the support of the test function f contained in $\mathcal{O}$. These algebras are endowed with a $*$-operation by defining $F(f)^* := F(f)^\dagger|_{\mathcal{D}}$. A definition of a $*$-algebra is given in the appendix.

The *local commutativity axiom* stated next enforces the Bose-Fermi alternative: The only particle statistics allowed are Bose-Einstein and Fermi-Dirac statistics.

Axiom E: For two test functions f, g with spacelike separated support

$$[F_i(f), F_j(g)]_\pm := F_i(f)F_j(g) \pm F_j(g)F_i(f) = 0 \tag{2.8}$$

holds for one of the signs.

Axiom F: The vacuum is cyclic for the algebra of field operators. This means that the set of vectors in $\mathcal{H}$ obtained from the action of all polynomials $P(F_1(f), F_2(g), \ldots)$ in the smeared field operators $F_1(f), F_2(g), \ldots$ is dense in $\mathcal{H}$. This set will be called $\mathcal{D}_0$.

These axioms formalise the essential properties any quantum field theory on Minkowski space is expected to have. The field operators may arise from the quantisation of a classical field theory, but this does not have to be the case.

The idea that measurements of observables in spacelike separated regions should be independent of each other and thus commute implies that observable fields must obey commutation relations. Green has shown [Gre53] that certain trilinear combinations of the usual creation and annihilation operators are compatible with the field equations. These lead to the possibility of more general statistics called *parastatistics*. In low spacetime dimensions the *braid group statistics* is another relevant statistics.

Axiom **F** implies that the smeared field operators form an irreducible set of operators in the Hilbert space $\mathcal{H}$. It takes some care to state this precisely, since we are dealing with unbounded operators. Irreducibility means that, if there is a bounded operator C satisfying $\langle \Phi, CF_i(f)\Psi\rangle = \langle F_i(f)^\dagger \Phi, C\Psi\rangle$ for all $F_i(f)$, then C is a multiple of the identity in $\mathcal{H}$.

Since quantum field theory merges quantum mechanics and special relativity, aspects of both theories enter the axioms. The universal cover of the Poincaré group appears in the discussion because one is interested in representations of the Poincaré group up to a factor, i.e. projective representations. It was shown by Wigner [Wig39] and Bargmann [Bar54] that there is a one-to-one correspondence of projective representations of the restricted Poincaré group and representations of its universal cover.

Other axioms may be added, for example, to allow the formulation of scattering theory. The possibilities include some *completeness condition* expressing that the field operators are sufficient to approximate any operator on $\mathcal{H}$ or the *time-slice axiom* stating that the fields at any time are determined by the fields in a small time-slice.

In [SW00, Str75] one can find a discussion showing the independence of these axioms. It is possible to construct free quantum field theories satisfying these assumptions, so the framework is consistent. Furthermore it is shown that some attempts of weakening these axioms in fact lead to theories satisfying the original axioms **A** – **F**. On the contrary, strengthening of the axioms can cause contradictions. In this sense, the framework is quite stable concerning modifications.

Nevertheless, in spite of examples in low dimensional spacetimes [GJ72, GJ81, BK04], it is not clear whether there is any consistent interacting theory in four spacetime dimensions which can be formulated in this framework. Despite of this drawback, the Wightman axioms have proven to be useful because they allowed for rigorously proving some fundamental theorems of quantum field theory rigorously. Among these are the PCT theorem presented in the following section, and the spin-statistics theorem. A discussion of the spin-statistics theorem is postponed to Section 2.3, where its development and present status will be reviewed.

2.1.1 The PCT Theorem

The full Poincaré group contains two types of discrete operations, one incorporating a reflection in time, T, and another type implementing a reflection in space, P.[4] A third discrete operation on an equation describing a charged particle is the charge conjugation C. The PCT theorem states that any local quantum field theory is invariant under the product of the three symmetries. The PCT theorem goes back to work of Schwinger [Sch51], Lüders [Lüd54], Pauli [Pau55], Bell [Bel55] and Jost [Jos57]. Its validity does not depend on the individual realisation of P, C and T. In fact, the famous experiment of Wu et al. [WAH$^+$57] proved in 1957 that parity invariance is violated in β-decays. Later in 1964 it was discovered by Christenson et

[4] Of course the combination of both is a third type.

al. [CCFT64] that also the combined symmetry CP is violated in weak interactions.

Here the PCT theorem is presented in its simplest version for a real scalar field.

Theorem 2.1 (The PCT theorem). *Let F be scalar quantum field satisfying the Wightman axioms. Then there exists a unique (up to a factor) antiunitary operator Θ in $\mathcal{H}$ with*

$$\Theta F(f)\Theta^{-1} = F(\hat{f}),$$

where $\hat{f}(x) = \overline{f(-x)}$ and the bar denotes complex conjugation.

A similar statement holds for general quantum fields with finitely many components and, as explained in [SW00], it can be strengthened to give necessary and sufficient conditions for the existence of Θ.

2.1.2 The Theorem of Bisognano and Wichmann

The results outlined in this section are another demonstration of the power of the axiomatic setup put forward by Wightman. Furthermore, they provided the starting point for a couple of interesting works, especially in the algebraic formulation of QFT. Some of these will be discussed in Section 2.2 because they, just as the theorem of Bisognano and Wichmann, also motivate the approach taken in this thesis.

In 1975 Bisognano and Wichmann discussed the duality condition for a scalar field based on the Wightman axioms [BW75]. The idea of duality conditions has its origin in the requirement that observables measured in spacelike separated regions are expected to commute.

If one considers the algebra $\mathbf{F}(\mathcal{O})$ of field operators located in a region $\mathcal{O}$, then the set of operators commuting with all operators in $\mathbf{F}(\mathcal{O})$ should contain the set of observables located in the causal complement of $\mathcal{O}$. Denoting, as introduced before, the causal complement of $\mathcal{O}$ by $\mathcal{O}'$, this statement would read

$$\mathbf{F}(\mathcal{O})' \supset \mathbf{F}(\mathcal{O}'). \tag{2.9}$$

Strengthening this relation to an equality,

$$\mathbf{F}(\mathcal{O})' = \mathbf{F}(\mathcal{O}'), \tag{2.10}$$

is called a duality. Since the field operators are unbounded so that the commutant has to be defined carefully, the relations (2.9) and (2.10) have to be considered as formal ones in this context. They cannot hold in this form, because the commutant of an unbounded operator is defined to be the set of all *bounded* operators commuting with all spectral projections of the unbounded operator. Thus the commutant is a set of bounded operators, whereas $\mathbf{F}(\mathcal{O}')$ is a set of unbounded ones.

Bisognano and Wichmann constructed von Neumann algebras[5] of bounded operators associated with the field algebras of the right and left wedge which satisfy

[5] For a definition see Section 2.2.1.

the duality condition. Their investigation led to some other very interesting results which will be of importance in motivating the work presented in this thesis.

Consider the one-parameter group $(\Lambda_R(t))_{t\in\mathbb{R}}$ of pure Lorentz transformations (boosts) which leaves the wedge W_R invariant (eq.(2.2) on p.10) and denote its representation on the Hilbert space $\mathcal{H}$ by

$$V(t) = U(0, \Lambda_R(t)) =: e^{-iKt}. \tag{2.11}$$

Here K is the (unbounded) self-adjoint generator of the one-parameter group $V(t)$ according to the theorem of Stone ([RS80, Theorem VIII.8]). The one-parameter group $V(t)$ can be continuously extended to the strip $\{z \in \mathbb{C}; 0 \le \mathrm{Im}(z) \le \pi\}$ so that it is analytical in its interior. For any $C \in \mathbf{F}(W_R)$ the element $C\Omega$ is in the domain of $V(i\pi)$. And if Θ denotes the PCT operator and $\rho_R(\pi)$ the rotation by the angle π which leaves the right wedge invariant, then

$$U(0, \rho_R(\pi))\Theta\, V(i\pi)C\Omega = C^*\Omega. \tag{2.12}$$

The PCT operator is antiunitary, so $J := U(0, \rho_R(\pi))\Theta$ is antiunitary, too. Let j be the reflection by the edge of the wedge W_R. The following properties can be verified directly,

$$J^2 = \mathbb{1}, \qquad J\Omega = \Omega, \qquad JU(a,\Lambda)J = U(ja, j\Lambda j). \tag{2.13}$$

Equation (2.12) is closely related to the modular theory of Tomita and Takesaki which is the subject of Section 2.2.1. The operators $V(t)$ and J act on the field algebra $\mathbf{F}(W_R)$ in a geometrical way, namely

$$V(t)\mathbf{F}(W_R)V(t)^\dagger = \mathbf{F}(W_R),$$
$$J\mathbf{F}(W_R)J = \mathbf{F}(W_L).$$

The shorthand

$$\Delta^{1/2} := V(i\pi) = e^{\pi K}$$

will be used for the positive operator $V(i\pi)$. In a subsequent paper, Bisognano and Wichmann generalised the analysis to charged quantum fields [BW76]. Essentially, the results stay the same in this case. The main difference is that now also fermionic fields have to be included. This implies that one has a unitary representation of the universal cover of the Poincaré group and that anticommutation relations are allowed. The role of the one-parameter group $(\Lambda_R(t))_{r\in\mathbb{R}}$ is played by its (unique) lift to a one-parameter group in the double cover of the Lorentz group.

2.1.3 Generalising the Wightman Axioms

The PCT theorem and the results of Bisognano and Wichmann are examples for structural insights gained from the axiomatic setup, and the spin-statistics theorem is another one. But of course it is natural to check whether this axiomatic setup can be further refined or improved. We already mentioned the article [Str75] and the book [SW00] in which the mutual interplay of the axioms and possible modifications are discussed.

The assumption which may have the weakest support by physical arguments is the requirement of the field to have only finitely many components. There is up to now no experimental evidence for fields with infinitely many components, but they are not "excluded" by basic physical principles. And in algebraic quantum field theory they are not excluded in the formulation of the axioms.

The Wightman formalism has been generalised by Streater [Str67] accordingly to admit quantum fields with infinitely many components. He replaced assumption **(D)** by

Assumption (D′). There is a Hilbert space $\mathfrak{C}$, the component space, carrying a representation $\tilde{D}$ of the universal cover of the Lorentz group. A quantum field F is a map from $\mathcal{S} \times \mathfrak{C}$ to the unbounded operators in $\mathcal{H}$ with

$$U(a, A)F(\mathfrak{c}, f)U(a, A)^{\dagger} = F(\tilde{D}(A)\mathfrak{c}, f_{\{a,A\}}).$$

Recall that U was the representation of the universal cover of $\mathcal{P}_+^{\uparrow}$ from assumption **A** and that $f_{\{a,A\}}(x) = f(\tilde{\lambda}(A^{-1})x - a)$ with the covering map $\tilde{\lambda}$.

This generalisation has stronger consequences than one may expect. In fact it renders both the spin-statistics theorem and the PCT theorem invalid. In [Str67], presenting the generalisation to infinitely many components, Streater constructed an example of a quantum field with infinitely many components which exhibits the wrong connection between spin and statistics. The reason for this is the "wrong" choice of the representation under which the quantum field transforms. In [OT68] Oksak and Todorov defined a quantum field with infinitely many components which does not admit a PCT operator Θ. Some more details on fields transforming under infinite-dimensional representations of the universal cover of the Lorentz group can be found in [BLOT90, Chapter 9 and its appendix].

These counterexamples have an important consequence. They show that, if one wants to generalise Wightman's framework to fields with an arbitrary number of components and at the same time keep the axioms powerful enough to prove general structural theorems, then additional assumptions have to be introduced. These assumptions have to be physically well-motivated, otherwise the modification of the successful framework does not seem to be justified.

This is the point where the Bisognano-Wichmann theorem enters the stage. As explained in Section 2.1.2, this theorem holds for a quantum field satisfying the Wightman axioms and relates the so-called modular objects J and Δ^{it} associated with field algebras of wedges to the geometric transformations of reflections and pure Lorentz transformations. Since these properties are consequences of the original axioms, assuming one or both of them would be a sensible generalisation of the original setup, if, at the same time, one drops other assumptions. This idea was successfully applied in [Kuc05] where the geometrical action of the modular group was assumed. For a theory with possibly infinitely many components a unitary representation of the double cover of the rotation group was constructed by Kuckert for which the spin-statistics theorem and the PCT theorem hold. This analysis will be presented in Section 3.

2.2 The Relevance of Modular Theory for Quantum Field Theory

Modular theory is one of the examples where independent developments in mathematics and physics suddenly are discovered to be closely related. On the mathematical side, modular theory emerged in the study of von Neumann algebras with a cyclic and separating vector[6] by Tomita. Further important contributions by Takesaki led to the alternative name *Tomita-Takesaki theory*. A brief introduction to this theory can be found in [BR79], comprehensive treatments are [Tak02] and its subsequent volumes by Takesaki or [KR97a, KR97b].

In physics similar concepts appeared in the work of Haag, Hugenholtz and Winnink [HHW67] on equilibrium states in quantum statistical mechanics. In 1967 the connection between the two approaches was recognised, and this triggered fruitful research activities. After a brief description of the foundations of modular theory some of its relations to physics will be indicated.

These relations corroborate the point of view that the Bisognano-Wichmann properties are not only consequences of the Wightman axioms but natural features of Wightman quantum fields with infinitely many components and of algebraic quantum field theory. An extensive discussion of applications of modular theory in quantum field theory is contained in [Bor95, Bor00].

2.2.1 Modular Theory

Modular theory (or Tomita-Takesaki theory) yields some deep theorems in the mathematical field of von Neumann algebras. So first some basic definitions and properties of von Neumann algebras have to be introduced. Let $\mathcal{L}(\mathcal{H})$ be the algebra of bounded linear operators mapping the Hilbert space $\mathcal{H}$ into $\mathcal{H}$, and let A^* denote the adjoint operator of $A \in \mathcal{L}(\mathcal{H})$.

The commutant of a subset $\mathfrak{M}$ of $\mathcal{L}(\mathcal{H})$ is the set of operators in $\mathcal{L}(\mathcal{H})$ commuting with all elements in $\mathfrak{M}$. Usually the commutant of $\mathfrak{M}$ is denoted $\mathfrak{M}'$ and the double commutant $(\mathfrak{M}')'$ by $\mathfrak{M}''$. A von Neumann algebra is a $*$-subalgebra $\mathfrak{M}$ of $\mathcal{L}(\mathcal{H})$

[6] See definition 2.1 on the following page.

which coincides with its double commutant,[7] i.e. $\mathfrak{M} = \mathfrak{M}''$. In the following, $\mathfrak{M}$ will always denote a von Neumann algebra of operators acting on the Hilbert space $\mathcal{H}$.

Definition 2.1 (Cyclic and separating vector). *A vector Ω in $\mathcal{H}$ is called* cyclic *for the von Neumann algebra $\mathfrak{M}$ if the set $\mathfrak{M}\Omega$ is dense in $\mathcal{H}$. Ω is* separating *for $\mathfrak{M}$ if for all $A \in \mathfrak{M}$ $A\Omega = 0$ implies $A = 0$.*

Now let Ω be a cyclic and separating vector for the von Neumann algebra $\mathfrak{M}$. Then it can be proven that Ω is also cyclic and separating for $\mathfrak{M}'$. Define antilinear operators S_0 and F_0 by

$$\begin{aligned} S_0 &: \mathfrak{M}\Omega \to \mathcal{H}; \quad A\Omega \mapsto A^*\Omega, \\ F_0 &: \mathfrak{M}'\Omega \to \mathcal{H}; \quad B\Omega \mapsto B^*\Omega. \end{aligned} \tag{2.14}$$

S_0 and F_0 are closable (see proof of Proposition 2.4 on page 25), and hence there exists a unique polar decomposition[8] of the closure S of S_0 into the product of an antiunitary operator J and a positive operator $\Delta^{1/2}$. J is called modular conjugation, and Δ is the modular operator associated with $(\mathfrak{M}, \Omega)$.

The following proposition collects some properties of S and the closure F of F_0.

Proposition 2.2. *Let S and F be the closures of S_0 and F_0, respectively, and let $S = J\Delta^{1/2}$ be the polar decomposition of S. Then*

$$\begin{aligned} \Delta &= FS, & \Delta^{-1} &= SF, \\ S &= J\Delta^{1/2}, & F &= J\Delta^{-1/2}, \\ J &= J^*, & J^2 &= 1, \\ \Delta^{-1/2} &= J\Delta^{1/2}J. \end{aligned}$$

Proof. See [BR79, Proposition 2.5.11]. □

Since Δ is self-adjoint and positive, one can define the bounded operator Δ^{it} for $t \in \mathbb{R}$ by use of the spectral theorem for unbounded normal operators.[9] The deep result of Tomita and Takesaki is

Theorem 2.2 (Tomita-Takesaki theorem). *Let $\mathfrak{M}$ be a von Neumann algebra with a cyclic and separating vector Ω and associated modular conjugation J and modular operator Δ. Then there holds*

$$J\mathfrak{M}J = \mathfrak{M}' \tag{2.15}$$

and

$$\Delta^{it}\mathfrak{M}\Delta^{-it} = \mathfrak{M}. \tag{2.16}$$

[7] A $*$-subalgebra $\mathfrak{M}$ fulfils $\mathfrak{M}^* = \mathfrak{M}$.

[8] For some details concerning the polar decomposition of closed operators see [RS80, Chapter VIII.9].

[9] This is treated in [RS80, Chapter VIII.3].

The modular automorphism group associated with the pair $(\mathfrak{M}, \Omega)$ is the one-parameter group $(\sigma_t)_{t\in\mathbb{R}}$ given by

$$\sigma_t(A) = \Delta^{it} A \Delta^{-it} \qquad \text{for } A \in \mathfrak{M}, t \in \mathbb{R}. \tag{2.17}$$

It is weakly continuous and provides an essential tool in the classification of von Neumann algebras, e.g. in the classification of the factors of type III [Con73] which are those typically occurring in quantum field theory (see [Yng05] and references therein).

2.2.2 Algebraic Quantum Field Theory

Besides the Wightman axioms there have been other attempts to put quantum field theory on a rigorous basis. A different approach, called *algebraic quantum field theory*, was formulated by Haag, Araki and Kastler in [Haa57, Ara62, HK64]. An excellent reference is the monograph [Haa96]. In algebraic quantum field theory (AQFT) the focus is on the algebras of observables located in spacetime regions without referring to a particular Hilbert space. There are several arguments in favour of this point of view. For finite-dimensional physical systems the Stone-von Neumann uniqueness theorem guarantees that all representations of the position and momentum operators satisfying the usual commutation relations are unitarily equivalent. So the physical properties do not depend on the representation of the operators. This uniqueness theorem breaks down for systems with infinitely many degrees of freedom like the ones described by quantum field theories. Shifting the focus from the representations to the algebraic properties of the observables circumvents the problem of inequivalent representations.

Another argument has its origin in the existence of the so-called *Borchers class* of fields. If one has a Wightman field theory based on a field F, then there are several other fields, the fields *relatively local to* F, which all lead to the same scattering matrix and therefore describe the same physics. Furthermore, in the Haag-Ruelle collision theory the essential ingredient are not the field operators themselves, but the correspondence between spacetime regions $\mathcal{O}$ and the algebras of operators located in $\mathcal{O}$. In [Haa96, Chapter III.1] Haag refers to a field being a "coordinatization" of this correspondence.

Finally, the shift of the focus away from unobservable fields like fermionic fields, gauge fields and other unobservable quantities to observable quantities seems very reasonable. These ideas lead to the following axiomatic setup. In the algebraic formulation a quantum field theory is given by a *net of observables*. This is a map

$$\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O}), \qquad \mathcal{O} \subset \mathbb{R}^{1+3}, \tag{2.18}$$

of open regions $\mathcal{O}$ in Minkowski space to C*-algebras $\mathfrak{A}(\mathcal{O})$. Observables are self-adjoint elements in the algebras $\mathfrak{A}(\mathcal{O})$. This net satisfies isotony: if $\mathcal{O}_1 \subset \mathcal{O}_2$, then

$\mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2)$. Poincaré invariance translates to the requirement of automorphisms $\alpha_g, g \in \mathcal{P}_+^\uparrow$, implementing Poincaré transformations on the net via

$$\alpha_g \mathfrak{A}(\mathcal{O}) = \mathfrak{A}(g\mathcal{O}), \quad g \in \mathcal{P}_+^\uparrow. \tag{2.19}$$

The principle of *locality* is encoded in the usual way by the requirement of local observables in spacelike separated regions to commute. In this setting, a state is a continuous positive linear functional of norm 1. The investigation of states on the abstract algebras is related to the study of the representations, because every state ω defines a representation π_ω in some Hilbert space via the GNS-construction (see appendix). By considering the double commutant $\pi_\omega(\mathfrak{A}(\mathcal{O}))''$ one gets a von Neumann algebra and has the objects and theorems of Tomita-Takesaki theory at hand. Conversely, any nontrivial vector Ψ in the Hilbert space of a representation π defines a state on the net via

$$\omega(A) = \frac{1}{\langle \Psi, \Psi \rangle} \langle \Psi, \pi(A)\Psi \rangle. \tag{2.20}$$

In the analysis of the representations one imposes physically motivated selection criteria. Doplicher, Haag and Roberts studied states which, outside of a bounded region, look like vacuum states [DHR69a, DHR69b]. Buchholz and Fredenhagen extended the analysis to states localisable in spacelike cones [BF82]. Under this assumption unobservable fields like fermionic fields and the gauge group show up in the field algebras constructed from the algebras of observables [DHR69a, DHR69b, DHR71, DHR74, DR90]. For both selection criteria the field algebra and gauge group construction are unique [DR90]. An important result is that the possible statistics for the field algebra are restricted to para-Bose and para-Fermi statistics, and normal commutation relations hold. In 1+2 dimensions the Buchholz-Fredenhagen approach leads to braid group statistics. An interesting feature of the algebraic setting is that fields with infinitely many components are not ruled out by assumption. For some more recent reviews of the status of algebraic quantum field theory see [Buc00, FRS06].

2.2.3 Modular P_1CT Symmetry, Modular Covariance and the Unruh Effect

For finite-dimensional quantum statistical systems a satisfactory description of equilibrium is given for the canonical ensemble. An equilibrium state ω_β with inverse temperature β is a state given by a density matrix $\rho_\beta = 1/Z \exp(-\beta H)$, where H is the Hamiltonian of the system and $Z = \mathrm{Tr}(\exp(-\beta H))$. The one-parameter group $\mathbb{R} \ni t \mapsto \alpha_t(A) = \exp(itH) A \exp(-itH)$ yields the time evolution of the observable A in the Heisenberg picture. The state ω_β has the property

$$\omega_\beta(\alpha_t(A)B) = \omega_\beta(B\,\alpha_{t+i\beta}(A)). \tag{2.21}$$

In the seminal article [HHW67] Haag, Hugenholtz and Winnink proved that equation (2.21) essentially describes equilibrium states with inverse temperature β for infinite systems. A simple calculation (see Chapter V.2 of [Haa96]) shows that the modular group associated with the state ω satisfies the KMS-condition for $\beta = -1$. This is an important example for the relevance of modular theory in physics. The dynamics of an equilibrium state is exactly given by the modular automorphism group.

There is another setting in which the modular group shows its significance. In 1976, Unruh [Unr76] investigated the question of black hole evaporation which was discovered by Hawking [Haw75]. An important result is that a uniformly accelerated observer in flat Minkowski space perceives the vacuum as a thermal heat bath with temperature $T = \frac{a}{2\pi c k_b}$. Here c is the speed of light, k_b is Boltzmann's constant and a the acceleration of the observer. Sewell elaborated this in the framework of Wightman quantum field theory [Sew80, Sew82] and pointed out the close connection of this result with the Bisognano-Wichmann theorem discussed in Section 2.1.2, with the KMS-condition and with Tomita-Takesaki theory. The modular group of wedge regions in Minkowski space implements a one-parameter group of boosts [BW75, BW76]. A uniformly accelerated observer, described by a one-parameter group of boosts detects the vacuum state as an equilibrium state with nonzero temperature [Unr76, Sew80, Sew82]. Moreover, the dynamics of equilibrium states (KMS-states) coincides with the rescaled modular group associated with the state [HHW67].

By the Bisognano-Wichmann theorem, the modular conjugation implements a certain PCT symmetry, namely the reflection in charge, time and one spatial direction [BW75, BW76]. The same connection was shown to hold by Sewell [Sew82] in a different context. For a uniformly accelerated observer, the vacuum state looks like a KMS-state and the modular conjugation associated with this KMS-state turns out to be a $\mathrm{P}_1\mathrm{CT}$ operator, a reflection in time, charge and one spatial direction.

The results of Sewell have been obtained in the Wightman framework, but the work of Bisognano and Wichmann is also concerned with the algebraic formulation of QFT. They discuss various conditions under which one can associate with a Wightman theory a net of observables in the sense of Araki, Haag and Kastler. The key properties that Bisognano and Wichmann derived have later been coined *modular covariance* in [BGL95] and *modular P_1CT symmetry* in [Kuc95] in a different context.

Modular Covariance. The modular group of the right wedge implements boosts on the field algebras in the form

$$\Delta_{W_R}^{it}\mathbf{F}(\mathcal{O})\Delta_{W_R}^{-it} = \mathbf{F}(\Lambda_{W_R}(2\pi t)\mathcal{O}), \tag{2.22}$$

where Δ_{W_R} is the modular operator associated with the field algebra of the right wedge and $\Lambda_{W_R}(t)$ is the boost in x_1-direction with rapidity t defined in eq. (2.2).

Modular P_1CT symmetry. The modular conjugation J_W associated with the wedge algebra $\mathbf{F}(W)$ acts geometrically as the product of a charge conjugation, reflection in time and reflection in one spatial direction. This operation is called P_1CT conjugation.

It is an interesting question whether modular covariance or modular P_1CT symmetry hold in algebraic quantum field theory without an underlying Wightman theory for the net of observables. This would be natural since modular theory fits perfectly the von Neumann algebras as the building blocks of AQFT.

For two-dimensional theories a positive answer has been given by Borchers. He showed that irrespective of the dimension, the modular unitaries associated with a wedge have the correct commutation relations with the translations leaving the wedge invariant. Assuming wedge duality in 1+1 dimensional theories this allows for defining an extension of the net of observables. For this net the unitary operators $U(\Lambda(t)) := \Delta^{it}$ generate a covariant representation of the Poincaré group [Bor92, Theorem III.1]). It satisfies modular covariance by definition.

To prove the Bisognano-Wichmann property in higher dimensions requires to find the commutation relations between different modular groups, which is an unsolved problem as yet. For conformal theories the Bisognano-Wichmann properties have been successfully studied under various assumptions [HL82, BGL93, FG93, FJ96]. In general, it is nevertheless possible to construct counterexamples, where the modular objects associated with wedge regions do not have the expected behaviour [Yng94]. These are not covariant under the Poincaré group or do not obey wedge duality.

Another result is that the modular objects act as expected, if they act geometrical in a very general sense [Kuc01] and map algebras associated with open sets to algebras associated with open sets. All these results suggest that a geometrical action of the modular objects in any AQFT would be natural. So it was tried to change the strategy of deriving modular covariance or modular P_1CT-symmetry and rather assume one of these (or both [FM91]) to hold and to study the consequences thereof.

As explained above, modular covariance essentially is the same as the Unruh effect. This profound physical explanation gives further justification to include it as an assumption in algebraic quantum field theory.

In a series of papers, Guido et al. [GL92, BGL93, BGL95, GL95, GL96, GLRV01] assumed modular covariance to hold and were able to prove the following consequences. One can construct a covariant representation of the Poincaré group from the modular unitaries Δ^{it} for $t \in \mathbb{R}$. In even dimensions there exists a PCT operator under which the theory is invariant. Furthermore, irrespective of the number of dimensions, the spin-statistics theorem holds. This shows that modular covariance is a physically well-motivated assumption, powerful enough to establish numerous results. Moreover, it is an appropriate condition to rule out the counterexamples of Streater [Str67] and Oksak and Todorov [OT68] to the spin-statistics theorem and the PCT theorem for fields with infinitely many components.

Besides modular covariance defined in equation (2.22), the Bisognano-Wichmann theorem suggests a geometric action of the modular conjugation on the net given by

$$J_{W_R}\mathbf{F}(\mathcal{O})J_{W_R} = \mathbf{F}^t(j_{W_R}\mathcal{O}). \tag{2.23}$$

Here j_{W_R} is the reflection in the x_2-x_3 plane in Minkowski space which constitutes the edge of the right wedge. This property has been coined *modular P_1CT symmetry*. In [GL95] it was shown that modular covariance entails modular $\mathrm{P_1CT}$ symmetry.

In the series of papers [BS93, BDFS00, BFS99, BS04, BS05] Buchholz, Summers and collaborators formulated and investigated a *principle of geometric modular action* to single out physically distinguished states on general spacetimes. Another aim is to extract information about the geometrical structure of spacetime from the algebras of observables and states instead of postulating it.

The basic ingredient is a generalised version of modular $\mathrm{P_1CT}$ symmetry. It is the postulate that the modular conjugation should map every algebra of observables located in an open spacetime region *region* again to an algebra located in an open region. In [BS93] the modular conjugations have been used to construct a representation of the translation subgroup of the Poincaré group. In Minkowski space, states which fulfil the principle of geometric modular action indeed are vacuum states. Therefore the principle of geometric modular action seems to be a reasonable selection criterion, and subsequently it has been applied to theories in de Sitter space, anti-de Sitter space and Robertson-Walker spacetimes in [BFS99, BFS00, BDFS00, BMS01].

In [BDFS00] the representation of the translation group mentioned above has been extended to a representation of the Poincaré group. This construction was simplified considerably in [BS04]. The result obtained is

Proposition 2.3. *Let J be a continuous map from the set of reflections $\mathcal{R} \subset \mathcal{L}_+$ in edges of wedges into an arbitrary topological group $\mathcal{J}$. If J satisfies $J(\lambda)^2 = \mathbb{1}$ and $J(\lambda)J(\lambda_1)J(\lambda) = J(\lambda\lambda_1\lambda)$ for $\lambda, \lambda_1 \in \mathcal{R}$, then J is the restriction to $\mathcal{R}$ of a continuous homomorphism mapping $\mathcal{L}_+$ into $\mathcal{J}$.*

Proof. See [BS04, Proposition 2.8]. □

The topological group $\mathcal{J}$ will be interpreted as the group generated by the modular conjugations associated with von Neumann algebras of observables. If one transfers this statement to the Wightman setting, then this representation is a representation of the Lorentz group and as such would describe bosonic fields. To obtain a description of fermionic fields one needs a representation of the cover of the Lorentz group. This is exactly the idea put forward in [Kuc05], where as a first step a representation of the universal cover of the rotation group was constructed using some of the arguments of [BS04]. This construction has been extended by Kuckert and the present author in [KL07] to a representation of the universal cover of the Lorentz group. In Chapter 4 this analysis will be set forth and it will be completed in Chapter 5, where a representation of the double cover of the Poincaré group is defined.

At this point it may be in order to summarise the essence of the section. The Bisognano-Wichmann properties, modular covariance and modular P_1CT symmetry, hold for a Wightman quantum field theory with a finite number of components and in two-dimensional AQFT as well as in conformal AQFT. There are partial results for higher-dimensional AQFT. The correct commutation relations for the modular unitaries and the translations hold. And there is the result of Kuckert [Kuc01, Kuc98b] stating that, if the modular objects act geometrically in a very general sense, then they act in the "correct" way. Furthermore, the modular objects are closely related to other physical settings like the description of equilibrium states or the Unruh effect.

On the other hand, assuming modular covariance and/or modular P_1CT symmetry allows for establishing important further results. Among these are the selection criterion for vacuum states, an algebraic PCT theorem and an algebraic spin-statistics theorem. Since a priori AQFT makes no restriction on the number of components of a quantum field, the last two examples show that modular covariance is an appropriate assumption to rule out the counterexamples of Streater, Oksak and Todorov. From the modular unitaries there has been constructed a unitary representation of the double cover of the Poincaré group, and one can define a unitary representation of the Poincaré group in terms of modular conjugations.

Modular P_1CT symmetry is a weaker or equivalent assumption compared with modular covariance, because Guido and Longo derived the former from the latter and because in [BDFS00] an example of a net of observables is given which satisfies modular P_1CT symmetry, but not modular covariance. Under an additional additivity assumption the conditions seem to be equivalent [Dav95].

Modular P_1CT symmetry suffices to obtain a spin-statistics theorem [Kuc05] for a rotationally invariant theory (see Chapter 3 in this thesis) and also for Lorentz- and Poincaré invariant theories (Chapter 4 and 5, respectively).

2.2.4 Modular Objects for Algebras of Unbounded Operators

The impressive results of the modular theory by Tomita and Takesaki have been achieved for von Neumann algebras of bounded operators. The case of algebras of unbounded operators is considerably less well understood. Even if one ignores the usual difficulties related to questions of domains for unbounded operators, problems already arise in formulating a sensible setup. For example, it is not a priori clear how to define the commutant of a set of unbounded operators. Some possible ways to deal with these problems and further references can be found in the monograph of Inoue [Ino98]. There the Wightman framework of quantum field theory is also discussed.

Fortunately, the deeper results of Tomita-Takesaki theory are not essential for the constructions performed in this thesis. Only the basic definitions of the modular objects and some simple properties are needed, and these can be formulated for the field algebras which are $*$-algebras of unbounded operators.

Let $(\mathcal{H}, \langle\cdot, \cdot\cdot\rangle)$ be a Hilbert space and let $\mathcal{A}$ be a $*$-algebra[10] of (unbounded) operators acting on a common dense domain $\mathcal{D}$ and mapping $\mathcal{D}$ into itself. Furthermore, let Ω be a cyclic vector for $\mathcal{A}$. For all $A \in \mathcal{A}$ we assume $A^* \subset A^\dagger$, where $(\cdot)^\dagger$ denotes the usual adjoint. Assume that there exists an algebra $\mathcal{A}'_\sigma$ commuting with $\mathcal{A}$ in the sense that for every $A \in \mathcal{A}$, $B \in \mathcal{A}'_\sigma$ and $\Phi, \Psi \in \mathcal{D}$ the relation $\langle AB\Phi, \Psi\rangle = \langle A\Phi, B^*\Psi\rangle$ holds. This is weaker than the usual notion of commutativity for unbounded operators in terms of commuting spectral projections. $\mathcal{A}'_\sigma$ has to be obey the same properties as $\mathcal{A}$, i.e. the operators are defined on the same dense domain $\mathcal{D}$, the $*$-operation is also the restriction of the adjoint to $\mathcal{D}$ and Ω is cyclic.

Define antilinear operators S and F on $\mathcal{A}\Omega$ and $\mathcal{A}'_\sigma\Omega$, respectively, by

$$\begin{aligned} S^0 A\Omega &= A^*\Omega, \\ F^0 B\Omega &= B^*\Omega. \end{aligned}$$

Proposition 2.4. *The operators S^0 and F^0 are closable.*

Proof. We show that $S^0 \subset (F^0)^\dagger$ by calculating for $A \in \mathcal{A}, B \in \mathcal{A}'_\sigma$

$$\begin{aligned} \left\langle B\Omega, S^0 A\Omega \right\rangle &= \langle B\Omega, A^*\Omega\rangle = \langle AB\Omega, \Omega\rangle \\ &= \langle A\Omega, B^*\Omega\rangle = \left\langle A\Omega, F^0 B\Omega \right\rangle \\ &= \left\langle B\Omega, F^{0\dagger} A\Omega \right\rangle. \end{aligned}$$

In the last equality the antilinearity of F^0 enters. So $F^{0\dagger}$ is densely defined, and S^0 is closable. The proof of F^0 being closable proceeds similarly. □

Denote the closures of S^0 and F^0 by S and F, respectively. As a closed operator S has a unique polar decomposition into an antiunitary operator J and a positive operator $\Delta^{1/2}$. J is called modular conjugation and Δ modular operator associated with $(\mathcal{A}, \Omega)$.

Proposition 2.5. *The modular objects J and $\Delta^{1/2}$ associated with a $*$-algebra of unbounded operators satisfying the assumptions stated above have the following properties:*

$$\begin{aligned} J &= J^\dagger, \\ J^2 &= 1, \\ J\Delta^{1/2}J &= \Delta^{-1/2}. \end{aligned}$$

[10] See appendix for a definition.

Proof. One has $S = S^{-1}$ because $S^0 = (S^0)^{-1}$. As a consequence $J\Delta^{1/2} = S = S^{-1} = \Delta^{-1/2}J^\dagger$. This entails $J\Delta^{1/2} = J^\dagger J\Delta^{-1/2}J^\dagger$, and the uniqueness of the polar decomposition yields $J = J^\dagger$ and $J\Delta^{-1/2}J = \Delta^{1/2}$. The statements now follow immediately. □

Now let $\mathcal{B}$ be a second algebra satisfying the same assumptions as $\mathcal{A}$ with the same cyclic vector Ω and the same domain $\mathcal{D}$. The modular operators $S_\mathcal{A}, S_\mathcal{B}$ and the modular objects will be labelled accordingly. If there is a unitary or antiunitary operator transforming the algebras into each other (and leaving Ω invariant), then it transforms also the modular objects into each other. This is well-known for the case of von Neumann-algebras and holds also for the $*$-algebras of unbounded operators defined above.

Proposition 2.6. *Let U be a unitary or antiunitary operator transforming the algebras into each other and leaving Ω invariant:*

$$U\mathcal{B}U^\dagger = \mathcal{A}.$$

Then $S_\mathcal{A} = US_\mathcal{B}U^\dagger$, $J_\mathcal{A} = UJ_\mathcal{B}U^\dagger$ and $\Delta_\mathcal{A} = U\Delta_\mathcal{B}U^\dagger$.

Proof. If $B \in \mathcal{B}$, then

$$S_\mathcal{B}B\Omega = B^*\Omega = US_a\underbrace{U^\dagger BU}_{\in\mathcal{A}}\Omega = US_\mathcal{A}U^\dagger B\Omega.$$

The statement now follows by uniqueness of the polar decomposition. □

These basic statements suffice to establish the modular operators and modular conjugations for the field algebras of unbounded operators occurring in the Wightman framework of quantum field theory.

2.3 The Spin-Statistics Theorem: Status and New Results

One of the most fundamental theorems of quantum field theory is the spin-statistics theorem. It connects two different properties of particles, the spin, labelling the unitary and irreducible representations of the Poincaré group, and the statistics. A field with components F_i obeys *Bose-Einstein* statistics if

$$[F_i(x), F_j(y)]_- = 0 \qquad \text{for } (x-y)^2 < 0, \tag{2.24}$$

and Fermi-Dirac statistics if

$$[F_i(x), F_j(y)]_+ = 0 \qquad \text{for } (x-y)^2 < 0. \tag{2.25}$$

Here $[\cdot, \cdot\cdot]_\pm$ denotes the usual commutator and anticommutator, namely

$$[F_i(x), F_j(y)]_\pm := F_i(x)F_j(y) \pm F_j(y)F_i(x).$$

The *spin-statistics theorem* states that a field with integer spin obeying *Fermi-Dirac* statistics necessarily is trivial, $F_i(x) \equiv 0$. Conversely, fields with half-integer spin have to be trivial if they obey Bose-Einstein statistics.

The spin-statistics theorem in this form is a kind of no-go theorem. If one wants to be very careful, one has to distinguish the spin-statistics theorem from the *spin-statistics connection* which is the *positive* statement that particles with half-integer spin obey the Fermi-Dirac statistics and particles with integer spin obey Bose-Einstein statistics.

If one assumes that Bose-Einstein statistics and Fermi-Dirac statistics are the only statistics allowed (the *Bose-Fermi alternative*), then the spin-statistics theorem states that nontrivial fields satisfy the spin-statistics connection. The Bose-Fermi alternative excludes other statistics as braid-group statistics (relevant in low dimensional models, see e.g. [FRS89, FG90]) and parastatistics [Gre53, GM65], occurring naturally in AQFT [DHR70, DHR71, DHR74, DR90].

The first ideas in the direction of a spin-statistics theorem arose in the 1936 paper of Pauli [Pau36]. He showed that one can quantise a scalar free field theory with commutation relations but runs into trouble assuming anticommutation relations. In the case of anticommutation relations, the charge densities at spacelike separated points do not commute. This is not acceptable for an observable quantity because of the causality principle.

In 1937, Iwanenko and Sokolow [IS37] investigated the quantisation of the Dirac equation. They showed that a quantisation with Bose-Einstein statistics always leads to negative energies for the particle or the antiparticle. Both papers lack mathematical rigour because of some formal reordering of factors.

In 1939, Fierz, as an assistant of Pauli, published a first proof of the spin-statistics theorem [Fie39]. He investigates a free field theory obeying locality, relativistic covariance and positivity of the energy. Additionally, it is assumed that a particle of arbitrary spin is described by an irreducible spinor. For particles with integer spin he gives a quantisation allowing for Bose-Einstein statistics, but leading to a contradiction for the case of Fermi-Dirac statistics. Similarly, particles with half-integer spin cannot be quantised if one assumes Bose-Einstein statistics, because the total energy is not positive.

Belinfante [Bel39] took another approach and derived the same result assuming invariance of the equations of motion under charge conjugation. Positivity of the energy then is a consequence. This work is in the spirit of the proof of Schwinger [Sch51] who derived the spin-statistics theorem from the PCT theorem more than ten years later.

In a subsequent publication [BP40] Belinfante and Pauli discussed some shortcomings of Belinfante's earlier paper. E.g., if one adds additional freedom and considers linear combinations of fields, then charge conjugation invariance is not sufficient to rule out Bose-Einstein statistics for fermionic particles. In this case the requirement of positive total energy, which was put forward by Pauli in his earlier paper, restores the result.

De Wet pointed out that Pauli's result established in 1936 is not correct. He treated the case of canonically quantised fields with spin 0, $\frac{1}{2}$ and 1 and gave a proof [dW40] of the spin-statistics theorem employing a Hamiltonian formulation.

The probably most widely recognised article on the spin-statistics theorem was published by Pauli [Pau40] who generalised and improved the work of Fierz cited above. The assumptions underlying the quantum field theory are the standard ones:

(i) Invariance of the theory under the restricted Lorentz group.

(ii) Existence of a state of lowest energy to be identified with the vacuum.

(iii) Locality: Physical observables commute at spacelike distances, and the fields commute or anticommute for spacelike separated arguments.

Pauli then considered a free theory of a general spinor field defined by a linear wave equation. He was able to divide the spinor representations of the proper Lorentz group into four classes according to their properties under PT transformations. This is quite close to the PCT theorem proven much later. Two of the classes correspond to the half-integer spin and the other two to the integer-spin case. Then, with arguments similar to those of Fierz, Belinfante and de Wet, he proves that integer spin fields cannot be quantised with anticommutation relations. For half-integer spin fields his argument of requiring the total energy to be positive is similar to the one of Fierz and Iwanenko and Sokolow.

All of these early proofs consider free theories and are model-dependent in respect to the fact that they are based on specific equations of motion like the Dirac equation or the Klein-Gordon equation. Furthermore, the argument used by Fierz, Belinfante, de Wet and Pauli to rule out anticommutation relations for integer spin contained some manipulations which were justified later by the Hall-Wightman theorem [HW57] used in the proofs of Lüders and Zumino and Burgoyne to be discussed in the next section.

The history of the spin-statistics connection is discussed in the book of Duck and Sudarshan [DS97] who also give excerpts of the relevant publications. Concerning this book, there exists an interesting review by Wightman [Wig99] commenting and clarifying some controversial statements of the book.

2.3.1 The Spin-Statistics Theorem in the Wightman Formalism

The endeavour in the 1950s to put quantum field theory on a rigorous base renewed the interest in the spin-statistics theorem and led to a couple of important new insights and results. The Wightman framework provided a chance to prove rigorous statements for interacting theories independent of a Lagrangian formulation, equations of motion or specific interactions.

Schwinger proved the spin-statistics theorem assuming PCT invariance in 1951 [Sch51]. This has to be compared with the work of Belinfante [Bel39] who assumed invariance under charge conjugation. Pauli comments on Schwinger's paper and reverses the arguments to derive the PCT theorem assuming the spin-statistics theorem to hold in [Pau55]. This mutual interplay of the PCT theorem and the spin-statistics theorem was resolved independently by the work of Lüders and Zumino [LZ58] (for the case of spin-0 and spin-$\frac{1}{2}$ fields) and Burgoyne [Bur58]. These authors achieved a general proof of the spin-statistics theorem for interacting quantum field theories from general physical principles. As a consequence, also the PCT theorem is proven from first principles.

The essential input in both papers is a theorem by Hall and Wightman [HW57] concerning the analyticity properties of the two-point functions $\langle \Omega, F(x)F(y)^*\Omega \rangle$. This theorem also justified some of the arguments used in the earlier proofs of Fierz, Belinfante, Pauli and de Wet. In the memorial volume [Jos60] dedicated to W. Pauli, Jost gives a nice summary of the status of the spin-statistics theorem in 1960 and presents the proofs of Pauli and Burgoyne.

The book of Streater and Wightman [SW00] has become the standard reference for the spin-statistics theorem in the Wightman formalism. There the proof of Burgoyne is presented along with all the necessary ingredients and an introduction to the required mathematical tools.

To summarise briefly, there is a satisfactory understanding of the spin-statistics theorem in the Wightman formalism. The input to the theory are natural properties of a quantum field theory, namely:

(i) Relativistic invariance.

(ii) The exclusion of negative energy states.

(iii) Locality, i.e. the requirement of fields evaluated at spacelike separated points to commute or anticommute.

(iv) Finite number of components of the field.

Then the spin-statistics theorem states that a nontrivial integer (half-integer) spin field cannot be quantised for a vanishing anticommutator (commutator) at spacelike separation. Additionally employing the Bose-Fermi alternative which excludes statistics different from Bose-Einstein or Fermi-Dirac, this yields the spin-statistics connection.

It is worth pointing out that no assumption is made on a specific form of equations of motion or on a possible interaction. The only postulate which seems to be less well supported by fundamental physical principles is the restriction on the number of components. Although there is no experimental observation requiring a field with infinitely many components as an explanation, there seems to be no general physical principle excluding this possibility. In Section 2.1.3 it has already been mentioned that dropping the assumption of a finite number of components has a serious impact on the validity of the PCT and the spin-statistics theorem. As discussed in Section 2.2, modular covariance and modular P_1CT symmetry are physically well-motivated candidates which may serve as a substitution to the postulate of a finite number of components. This will be demonstrated in Sections 3 and 4.

2.3.2 Spin and Statistics in Algebraic Quantum Field Theory

The question of the existence of a spin-statistics theorem in the algebraic formulation of quantum field theory suggests itself. Since AQFT makes no restriction on the number of components of a field, the counterexample of Streater [Str67] indicates that an additional assumption is required to prove the spin-statistics theorem in this setting.

Borchers proposed to consider only those representations of the observable algebra which admit the implementation of translation symmetry with the spectrum of the momentum operators contained in the closure of the forward light-cone. Then he showed [Bor65] that the Bose-Fermi alternative can always be accommodated in the setting without loss of generality. This work was implicitly based on assumptions violated in important models due to an error in older results.[11] But it inspired Doplicher, Haag and Roberts to their seminal series of papers [DHR69a, DHR69b, DHR71, DHR74].

Under the assumption of localisable charges, which unfortunately excludes long range forces like quantum electrodynamics, they proved the spin-statistics theorem. It is remarkable that the Bose-Fermi alternative is not an assumption, but a result in this setting. Only para-Fermi and para-Bose statistics of finite order occur. Using techniques developed by Epstein in a proof of the PCT theorem [Eps67], Buchholz and Epstein [BE85] generalised the result of Doplicher, Haag and Roberts to charges localised in spacelike cones. The results for low-dimensional theories where braid group statistics shows up, have already been mentioned [FRS89, FG90].

New results concerning the field algebra construction [DR90] and the proof of the PCT theorem in two-dimensional theories based on modular properties of quantum field theory [Bor92, Flo98] paved the way for further investigations of the spin-statistics theorem in a more algebraic spirit. The fruitful interplay of modular theory and AQFT has been outlined in Section 2.2.3. In that section, the series of papers [GL92, BGL93, BGL95, GL95, GL96, Gui95] by Brunetti, Guido and Longo has already been mentioned. It contains basically three versions of the spin-statistics

[11] See p.152 in [Haa96] and comment on p.2 in [DHR69a].

theorem: one for conformal field theory [GL96], one for four-dimensional spacetime [GL95] and one for the low-dimensional case[Gui95].

The conformal spin-statistics theorem can be derived just from basic principles of AQFT, because the Bisognano-Wichmann properties (modular covariance and modular P_1CT symmetry) hold in conformal AQFT due to the results of Brunetti, Guido and Longo [BGL93].

In four-dimensional AQFT it is again necessary to introduce a condition to eliminate the counterexample of Streater [Str67]. Guido and Longo prove that modular covariance is an appropriate condition for this goal. What is more, they show that modular covariance implies modular P_1CT symmetry, which is therefore the weaker of both properties.

Modular P_1CT symmetry was the starting point for Kuckert to prove the spin-statistics theorem in AQFT [Kuc95, Kuc98b, Kuc98a]. Since modular P_1CT symmetry implies *wedge duality*,[12] the Doplicher-Haag-Roberts field construction can be carried through. In these works the additional assumption of a compact gauge group is needed which enforces the uniqueness of the representation of the Poincaré group. This allows for lifting the modular P_1CT symmetry to the field system and finally yields a simple proof of the spin-statistics theorem.

Either of the approaches of Guido and Longo, starting from modular covariance, and of Kuckert, assuming modular P_1CT symmetry, has its advantages. Modular covariance is a stronger assumption than modular P_1CT symmetry. But in the work of Kuckert modular P_1CT symmetry alone did not suffice, a compact group of symmetries had to be assumed in addition.

As remarked above, Guido and Longo derived modular P_1CT symmetry from modular covariance. The equivalence of modular covariance and modular P_1CT symmetry has been studied in [Dav95] under an additional assumption of additivity of the net. But in [BDFS00, Chapter 5.3] there is an example of a net of observables satisfying modular P_1CT symmetry and violating modular covariance.

All results discussed so far are constrained to the case of flat Minkowski space. There are some results for the spin-statistics theorem in quantum field theory on curved spacetimes [GLRV01, Ver01], but we refer to the original literature, because the results in this thesis are based on flat Minkowski space.

2.3.3 Statement of Assumptions and Results

In the previous sections the historic and modern approaches to the spin-statistics theorem have been presented. The interplay between the spin-statistics theorem and the PCT theorem or invariance under time reversal, parity or charge conjugation alone is remarkable. The most general proof of the spin-statistics theorem in the Wightman framework of quantum field theory given by Burgoyne [Bur58]

[12] Wedge duality is the condition $\mathcal{A}(W)'' = \mathcal{A}(W')'$ for every wedge W.

resolves the mutual dependence between the PCT symmetry and the spin-statistics connection. Both follow from general principles of quantum field theory. The only assumptions which do not mirror basic physical considerations are the assumption of a finite number of components of the quantum field and the assumption of the Bose-Fermi alternative.

But eliminating the former of these leads to the counterexamples of Streater [Str67] to the spin-statistics connection and Oksak and Todorov [OT68] to the PCT theorem. So one may ask for a replacement of the assumption of a finite number of components by a physically better motivated one.

In AQFT, the Bose-Fermi alternative is not put in by hand, but follows for the field algebras. The only input are the algebras of observables and a suitable selection criterion for the physical states of the theory. Moreover, AQFT imposes no restriction on the number of components of a quantum field. Thus, also in this setting, there is a need for a physically well-motivated assumption to rule out the counterexamples of Streater and Oksak and Todorov mentioned above.

An interesting candidate for such an assumption can be built upon the theorem of Bisognano and Wichmann [BW76, BW75] (see Section 2.1.2). It states that modular covariance and modular P_1CT symmetry are fundamental properties inherited by any Wightman quantum field theory (with a finite number of components). These properties are closely related to important physical phenomena like the Unruh effect [Unr76, Sew82] and PCT symmetry (see Section 2.2.3). Furthermore, they also hold in two-dimensional AQFT [Bor92, Flo98] and in conformal quantum field theory [FM91, BGL93].

Imposing these properties in AQFT yields important results and allows for establishing the PCT theorem [GL95] and the spin-statistics theorem [Kuc98b] as discussed in the last section.

A first result by Kuckert [Kuc05] indicated that the assumption of modular P_1CT symmetry may also be useful in the generalised Wightman framework admitting fields with infinitely many components. The modular conjugations associated with algebras located in specific wedge regions were used to construct a unitary representation of the universal cover of the rotation group in three dimensions under which the fields transform covariantly. An additional representation in the component space could also be constructed and exhibits the spin-statistics connection if normal commutation relations hold. These results are the topic of the following chapter, where a simplified analysis based on spin groups is presented.

The generalisation of this work to the Lorentz- and Poincaré-covariant case is the main topic of the present thesis. The proof in the Lorentz-covariant case has been published in [KL07]. The precise assumptions for a quantum field F, which have been used in [Kuc05, KL07] and on which the work in this thesis is based, are the following. Let again $\mathcal{S}$ be the space of test functions. The concrete choice is not important for the analysis, the space of infinitely often differentiable functions with compact support would be equally well-suited. Recall that $\mathcal{W}$ is the set of wedge regions in Minkowski space whose edge contains the origin.

Field algebras and component space A quantum field F is an operator mapping elements of a component space $\mathfrak{C}$ and test functions to unbounded, densely defined operators on some Hilbert space $\mathcal{H}$. The component space $\mathfrak{C}$ is a linear space, possibly infinite dimensional. Furthermore, the following properties are assumed.

(i) The component space is "free from redundancies". This means, that if $\mathfrak{c}, \mathfrak{d} \in \mathfrak{C}$ and $F(\mathfrak{c}, f) = F(\mathfrak{d}, f)$ for all $f \in \mathcal{S}$, then $\mathfrak{c} = \mathfrak{d}$.[13]

(ii) There is a dense set $\mathcal{D}$ which is contained in the domain of all operators $F(\mathfrak{c}, f)$ for $\mathfrak{c} \in \mathfrak{C}$ and $f \in \mathcal{S}$. The domain $\mathcal{D}$ is also contained in the domain of all adjoint field operators $F(\mathfrak{c}, f)^\dagger$. Furthermore, $\mathcal{D}$ is mapped into itself under all field operators and their adjoints. For every $\mathfrak{c} \in \mathfrak{C}$ and $\Phi, \Psi \in \mathcal{D}$ the map $\mathcal{S} \ni f \mapsto \langle \Phi, F(\mathfrak{c}, f)\Psi \rangle$ is a tempered distribution.

(iii) The $*$-algebra of field operators is generated by the operators $F(\mathfrak{c}, f)|_{\mathcal{D}}$ and $F(\mathfrak{c}, f)^\dagger|_{\mathcal{D}}$ with $*$-operation $F(\mathfrak{c}, f)^* := F(\mathfrak{c}, f)^\dagger|_{\mathcal{D}}$. The $*$-subalgebras of field operators located in a spacetime region $\mathcal{O}$ are denoted by $\mathbf{F}(\mathcal{O})$, if the support of the test-functions is contained in $\mathcal{O}$. The main cases of interest will be the algebras $\mathbf{F}(a)$ located in wedges $a \in \mathcal{W}$.

(iv) The field algebras of different wedges differ, i.e. $a \neq b$ implies $\mathbf{F}(a) \neq \mathbf{F}(b)$ for $a, b \in \mathcal{W}$.

Vacuum state The vacuum state Ω is contained in $\mathcal{D}$. It is cyclic for the algebras $\mathbf{F}(a)$ for any wedge region $a \in \mathcal{W}$.

Normal commutation relations There exists a unitary and self-adjoint operator k on $\mathcal{H}$ with $k\Omega = \Omega$ and with $k\mathbf{F}(a)k = \mathbf{F}(a)$ for all $a \in \mathcal{W}$. Define $F_\pm := \frac{1}{2}(F \pm kFk)$. If $\mathfrak{c}$ and $\mathfrak{d}$ are arbitrary elements of $\mathfrak{C}$ and if $f, h \in \mathcal{S}$ have spacelike separated supports, then

$$\begin{aligned} F_+(\mathfrak{c}, f)F_+(\mathfrak{d}, h) &= F_+(\mathfrak{d}, h)F_+(\mathfrak{c}, f), \\ F_+(\mathfrak{c}, f)F_-(\mathfrak{d}, h) &= F_-(\mathfrak{d}, h)F_+(\mathfrak{c}, f) \qquad \text{and} \\ F_-(\mathfrak{c}, f)F_-(\mathfrak{d}, h) &= -F_-(\mathfrak{d}, h)F_-(\mathfrak{c}, f) \end{aligned}$$

for all $\mathfrak{c}, \mathfrak{d} \in \mathfrak{C}$.

The involution k is the *statistics operator* and $F_\pm$ are the *bosonic* and *fermionic components*, respectively. Defining

$$\kappa := \frac{1 + ik}{1 + i} \qquad \text{and} \qquad F(\mathfrak{d}, h)^t := \kappa F(\mathfrak{d}, h)\kappa^\dagger, \tag{2.26}$$

[13]This may be compared with the similar assumption *b*) in [BW76, Section III] for the case of a field with finitely many components.

the normal commutation relations can be expressed by the simple formula

$$\left[F(\mathfrak{c}, f), F(\mathfrak{d}, h)^t\right] = 0 \tag{2.27}$$

for spacelike separated supports of f and h. This property is referred to as *twisted locality*. The notation is also used for the algebras of field operators located in a spacetime region $\mathcal{O}$, namely, $\mathbf{F}(\mathcal{O})^t := \kappa \mathbf{F}(\mathcal{O})\kappa^\dagger$. Now we give the crucial assumption of modular $\mathrm{P_1CT}$ symmetry. Recall that for $a \in \mathcal{W}$ the reflection in the edge of the wedge a is denoted by j_a and that the modular conjugation associated with the wedge algebra $\mathbf{F}(a)$ is denoted by J_a. The modular objects for $*$-algebras of unbounded operators have been defined in Section 2.2.4.

Modular $\mathbf{P_1}$CT symmetry For any $a \in \mathcal{W}$ there exists a linear involution C_a in $\mathfrak{C}$ with

$$J_a\, F(\mathfrak{c}, f)\, J_a = F^t(C_a\mathfrak{c}, \overline{j_a f}), \tag{2.28}$$

where $j_a f(x) := f(j_a x)$ and the bar denotes complex conjugation. The map $\mathcal{W} \ni a \mapsto J_a$ is continuous.

For the field algebras the condition implies

$$J_a\, F(\mathcal{O})\, J_a = F^t(j_a\mathcal{O}). \tag{2.29}$$

Note that no unitary and covariant representation of the universal cover of the Poincaré group and no spectrum condition are assumed. For every wedge algebra $\mathbf{F}(a)$ the associated Tomita-Takesaki operator S_a and therefore also J_a and Δ_a are well-defined. This follows because $\mathbf{F}(-a)^t$ commutes with $\mathbf{F}(a)$ by twisted locality and plays the role of $\mathcal{A}'_\sigma$ needed for the proof of Proposition 2.4 on page 25. One can show that Ω is not only cyclic but also separating for the wedge algebras $\mathbf{F}(a)$.

The following results will be presented for a quantum field on four-dimensional Minkowski space obeying these assumptions. The universal covering group of the Lorentz group can be described by pairs of wedges in two ways. One possibility is to define a suitable equivalence relation on pairs of wedges, which is motivated by the simpler case of the rotation group. Then the quotient space $\mathbf{G}_L$ is a realisation of $\tilde{L}_+^\uparrow$. The definition of the covering map from $\mathbf{G}_L$ to $L_+^\uparrow$ is very intuitive. Take a pair of wedges characterising an element of the covering group. This pair is mapped to the product of two reflections in their edges, which is a Lorentz transformation. The difficult tasks are to prove that the equivalence relation is indeed well-defined and to prove that the quotient space has the necessary properties of the universal cover. This is the content of Section 4.2.1.

The second approach is somewhat converse to the one mentioned above. It is based on spin groups, which are models of the universal cover of the rotation group and Lorentz group. In this case the realisation of the universal cover is given and the identification with pairs of wedges has to be found. For the example of the rotation group this is presented in Section 3.2.2 and for the Lorentz group in Section 4.2.3.

The connection with quantum field theory is achieved when the models for $\tilde{L}_+^\uparrow$ described above are employed to define a representation in terms of pairs of modular conjugations. It is proved that the product of two modular conjugations is independent of the pairs of wedges specifying the same element of the covering group. Furthermore, the product of four modular conjugations is equal to the product of two modular conjugations, and this reduction is compatible with the universal covering group. This means that, if for wedges $a, \dots, f$ the relation

$$(J_a J_b)\,(J_c J_d) = J_e J_f$$

holds, then the pair (e, f) specifies the element of the covering group which is the product of the elements characterised by (a, b) and (c, d).

These properties allow for defining a unitary representation of $\tilde{L}_+^\uparrow$ in terms of pairs of modular conjugations (Theorems 4.3 and 4.4). This representation acts covariantly on the field. As a consequence, a representation in the component space is obtained. These representations exhibit the correct spin-statistics connection.

In Chapter 5.1, the construction of the representation is extended to the Poincaré group. The spin-statistics theorem for this case seems to be the most general one obtained so far (Theorem 5.1). The assumption of modular $\mathrm{P_1CT}$ symmetry is somehow in the spirit of the proofs of Belinfante [Bel39], who assumed charge conjugation invariance, and Schwinger [Sch51], who assumed invariance under time reversal.

As an additional result a full PCT operator Θ can be obtained as the product of three modular conjugations [Kuc05, KL07], see Theorem 5.2. Finally, the statements are transferred to a net of field algebras in the algebraic framework in Section 5.3.

So two of the most important theorems of quantum field theory, the spin-statistics theorem and the PCT theorem can be proven from the assumptions stated above. No spectrum condition is required and no covariant representation of the Poincaré group is given from the outset. Instead a representation canonically associated with the quantum field is constructed employing the assumption of modular $\mathrm{P_1CT}$ symmetry. In this setup, the number of components of the quantum field is not restricted to be finite. It is therefore more general than the usual Wightman framework, for which modular $\mathrm{P_1CT}$ symmetry is a consequence by the theorem of Bisognano and Wichmann.

2.4 Orthogonal Groups, Reflections and Spin Groups

This section serves as a brief introduction to some mathematical topics needed in the sequel. In the remainder of this section V will denote an n-dimensional vector space over the real numbers. If a nondegenerate symmetric form g is defined on this vector space, then one can define the orthogonal group associated with (V, g). It is the set $O(V, g)$ of all linear maps from V to V preserving the bilinear form g. Since the bilinear form is symmetric and nondegenerate, it can be described in a basis by a symmetric real invertible matrix. This matrix can be diagonalised with entries ± 1 on the diagonal. The number of plus and minus signs is independent of the basis. A corresponding sequence of plus and minus signs is called signature. This allows for labelling the orthogonal groups over finite-dimensional real vector spaces by the number of plus and minus signs in the diagonalised matrix. As remarked in Section 2.1 the full Lorentz group with the Lorentz metric g is defined to have signature $(+,-,-,-)$ and can therefore be denoted by $O(\mathbb{R}^4, g)$ or $O(1,3)$. The Lorentz group in 4 dimensions has four connected components.

$L_+^\uparrow$: The connected component of the identity.

$L_+^\downarrow$: Transformations with determinant one which reverse the time direction.

$L_-^\uparrow$: Transformations with determinant minus one which do not reverse the time direction.

$L_-^\downarrow$: Transformations with determinant minus one which reverse the time direction.

Accordingly, $L_+^\uparrow$ coincides with the connected component of the identity of $O(1,3)$, which is denoted by $O(3,1)_0$ (or $SO(1,3)_0$ to emphasise the value of the determinant).

A reflection in a vector space (V, g) is a map leaving a subspace invariant and multiplying elements of the orthogonal subspace by -1. A hyperplane is an $(n-1)$-dimensional subspace of the vector space and can therefore be characterised by a vector orthogonal to it. The reflection by the hyperplane orthogonal to $v \in V$ is given by the formula

$$x \mapsto j_v(x) := x - 2\frac{g(v,x)}{g(v,v)}v. \tag{2.30}$$

A reflection in other subspaces will be specified by a set of hyperplane reflections given by orthogonal unit vectors in the subspace not invariant under the reflection. For a r-tuple $v := (v_1, v_2, \ldots, v_r)$ of mutually orthogonal unit vectors, the reflection in the subspace orthogonal to all $v_i, i = 1, \ldots, r$, is denoted by j_v and given by

$$j_v = j_{v_1} \cdots j_{v_r}. \tag{2.31}$$

A subspace of $\mathbb{R}^{1+3}$ is *spacelike*, if there exists an orthogonal timelike vector. A *lightlike subspace* is spanned by a nontrivial lightlike vector. If a subspace is neither spacelike nor lightlike, then it is called *timelike*. If two subspaces are given by $v = (v_1, v_2, \ldots, v_d)$ and $w = (w_1, \ldots, w_s)$ then

$$j_v j_w j_v = j_{j_v w}, \tag{2.32}$$

where $j_v w := (j_v w_1, \ldots, j_v w_s)$. For hyperplanes this is a simple calculation using equation (2.30), and the generalisation follows immediately. By modular $\mathrm{P_1CT}$

symmetry, modular conjugations are related to reflections in edges of wedges, and these are reflections in time and one spacelike direction, hence elements of $L_-^\downarrow$. So mostly the reflections relevant in the subsequent sections are reflections by two-dimensional spacelike subspaces. The characterisation of wedges by zweibeine fits perfectly the description of a reflection in the edge of a wedge. The reflection in the edge of the wedge W_ξ with $\xi = (t_\xi, x_\xi) \in Z$ is given by j_ξ, because the edge of W_ξ is the two-dimensional spacelike subspace orthogonal to t_ξ and x_ξ.

It is a classical result of Cartan and Dieudonné (see e.g. [Art57]) that elements of the orthogonal groups in n dimensions can be decomposed into reflections. For any element one needs at most n reflections in hyperplanes. The product of an even number of wedge reflections is an element of $L_+^\uparrow$, and with different methods Buchholz and Summers [BS04] and Ellers [Ell04] proved that any element of the restricted Lorentz group can be decomposed into the product of two wedge reflections. This issue will be investigated further in Section 4.

2.4.1 Spin Groups

There are various possibilities to describe the universal cover of the restricted Lorentz group $L_+^\uparrow \cong SO(1,3)_0$. In physics, the most common one is the group $SL(2,\mathbb{C})$. But faced with the problem of constructing a representation in terms of pairs of modular conjugations one is lead to study geometrical properties of the universal covering group of $L_+^\uparrow$ which are related to reflections. The group $SL(2,\mathbb{C})$ seems not to be appropriate for this purpose because reflections can only be put in by hand in an abstract extension and not in a natural and simple way.

To this end, a new realisation named $\mathbf{G}_L$ has been constructed in [KL07] by Kuckert and the present author. This model is based on pairs of reflections in edges of wedges and was used to define a representation which satisfies the spin-statistics relation and will be introduced in Section 4.2.1. It turns out that another model is equally well suited for this purpose, the spin group $Spin(\mathbb{R}^{1+3})_0$. This group and its connection to reflections will be introduced in the following section.

The definition of $Spin(\mathbb{R}^{1+3})_0$ is based on the *Clifford algebra* $Cl(V,g)$ associated with a finite-dimensional real vector space V which is equipped with a nondegenerate symmetric bilinear form g. The twofold covering groups $Spin(V,g)$ of the special orthogonal groups $SO(V,g)$ can naturally be described in this framework.[14] For details on the material covered in this section we refer to the references [LM89, Gal05].

Clifford algebra

Probably the most prominent example of a Clifford algebra occurring in physics is the algebra generated by the γ-matrices in the theory of the Dirac equation. These

[14] The twofold covering groups $Pin(V,g)$ of $O(V,g)$ are closely related.

are a set of 4×4-matrices $(\gamma_\mu)_{\mu=0,\dots,3}$ satisfying

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2 g_{\mu\nu} \mathbb{1}. \tag{2.33}$$

Here $g_{\mu\nu}, \mu, \nu = 0, \dots, 3$, are the components of the Lorentz metric in some given basis. These matrices generate a realisation of the Clifford algebra $Cl(\mathbb{R}^{1+3}, g)$ associated with Minkowski space and its metric. A certain subgroup of this Clifford algebra, $Spin(\mathbb{R}^{1+3})_0$, provides a model of the universal cover of the Lorentz group. Clifford algebras, named after their inventor William Kingdon Clifford, are a generalisation of the concept of complex numbers and quaternions. In the remainder of this section a general definition of Clifford algebras and spin groups associated with vector spaces with nondegenerate symmetric bilinear forms will be given.

Let V be a finite-dimensional vector space over $\mathbb{R}$ and g a nondegenerate symmetric bilinear form over V. Then a *Clifford algebra* $Cl(V, g)$ is an associative algebra with unit $\mathbb{1}$ together with a map

$$i : V \to Cl(V, g); \qquad (i(v))^2 = g(v, v)\mathbb{1}.^{15} \tag{2.34}$$

Furthermore, $Cl(V, g)$ is required to have the universal property that for every associative algebra $\mathcal{A}$ with unit $\mathbb{1}_\mathcal{A}$ and map $j : V \to \mathcal{A}$ satisfying $(j(v))^2 = g(v, v)\mathbb{1}_\mathcal{A}$ there is a unique algebra homomorphism $f : Cl(V, g) \to \mathcal{A}$ such that the diagram

$$\begin{array}{ccc} V & \xrightarrow{i} & Cl(V,g) \\ & {\scriptstyle j}\searrow & \downarrow{\scriptstyle f} \\ & & \mathcal{A} \end{array} \tag{2.35}$$

commutes. By this universal property any two Clifford algebras associated with (V, g) are isomorphic, justifying to speak of *the* associated Clifford algebra. One can prove that the definition of Clifford algebra is not empty. An example can be constructed by dividing the tensor algebra over V by a certain ideal [LM89] which represents the relations present in the Clifford algebra.

The universal property allows to extend arbitrary linear maps into any associative unital algebra, which obey the relation (2.34), to the whole Clifford algebra.

Proposition 2.7. *Let $f : V \to \mathcal{A}$ be a linear map into an associative unital algebra $\mathcal{A}$ satisfying $f(v) \cdot f(v) = g(v, v)\mathbb{1}_\mathcal{A}$. Then f extends* uniquely *to an algebra homomorphism $\tilde{f} : Cl(V, g) \to \mathcal{A}$.*

Corollary 2.8. *The involutive linear map $\alpha_0 : i(V) \to i(V); \quad i(v) \mapsto -i(v)$ extends uniquely to an involutive automorphism $\alpha : Cl(V, g) \to Cl(V, g)$.*

[15] Some authors prefer the convention $(i(v))^2 = -g(v, v)\mathbb{1}$.

Since V is naturally embedded into $Cl(V,g)$ via the injective map i, one denotes, by abuse of notation, also the elements $i(v), v \in V$, by v. We simply write vw for the multiplication of v and w in $Cl(V,g)$. If V is finite-dimensional and $\{e_1, \ldots, e_n\}$ is a basis of V, then $\{e_{i_1} \cdots e_{i_k}; 1 \leq i_1 < \ldots < i_k \leq n\}$ together with $e_0 := \mathbb{1}$ is a basis of $Cl(V,g)$. Ignoring the multiplicative structure, $Cl(V,g)$ is therefore a 2^n-dimensional (in our case real) vector space. As such it is isomorphic to $\mathbb{R}^{2^n}$ and carries a unique Hausdorff topology with respect to which addition and scalar multiplication are continuous (cf. [Bou89, IV.1.5]). The property $vv = g(v,v)\mathbb{1}$ implies $vw + wv = 2g(v,w)\mathbb{1}$, which is the defining relation (2.33) for the γ-matrices.

The covering groups $Pin(V,g)$ and $Spin(V,g)$

The groups $Spin(V,g)$ and $Pin(V,g)$ can be defined as specific multiplicative subgroups of $Cl(V,g)$. $Pin(V,g)$ is the subgroup of $Cl(V,g)$ generated by the (invertible) elements $v \in V$ with $g(v,v) = \pm 1$. An element $v \in V$ is invertible, if $g(v,v) \neq 0$. In this case the inverse is given by $v^{-1} = v/g(v,v)$. $Spin(V,g)$ is the subgroup of $Pin(V,g)$ generated by products of elements $v \in V$ with $g(v,v) = \pm 1$ with an even number of factors. The *twisted adjoint map* $\widetilde{Ad}$, which maps invertible elements $\phi \in Cl(V,g)$ to automorphisms of $Cl(V,g)$ via

$$\phi \mapsto \widetilde{Ad}_\phi \quad \text{with} \quad \widetilde{Ad}_\phi(v) = \alpha(\phi) v \phi^{-1}, \quad \text{with } \alpha \text{ from cor. 2.8.} \tag{2.36}$$

has a geometrical interpretation.

Proposition 2.9. *Let $v \in V$ with $g(v,v) \neq 0$. Then one has $\widetilde{Ad}_v(V) = V$ and furthermore*

$$\widetilde{Ad}_v(w) = w - 2\frac{g(v,w)}{g(v,v)}v \quad \textit{for } w \in V.$$

Proof. Let $v, w \in V$ with $g(v,v) \neq 0$. The inverse of v is given by $\frac{1}{g(v,v)}v$. Therefore, one computes

$$\begin{aligned} \widetilde{Ad}_v(w) &= \alpha(v) w v^{-1} = \frac{-1}{g(v,v)} vwv \\ &= \frac{-v}{g(v,v)}(2g(w,v) - vw) = w - 2\frac{g(v,w)}{g(v,v)}v. \end{aligned}$$ □

Comparing this result with equation (2.32) shows that for $v \in V$ with $g(v,v) \neq 0$ $\widetilde{Ad}_v$ is the reflection in the hyperplane orthogonal to v. As introduced above, we denote such reflections by j_v. Note that reflections in V preserve the quadratic form $g(\cdot,\cdot)$ and that $\widetilde{Ad}$ is a homomorphism. Hence it is a homomorphism from $Pin(V,g)$ and $Spin(V,g)$ into the orthogonal group $O(V,g)$ of the form g. Therefore, $\widetilde{Ad}$ restricted to $Pin(V,g)$ and $Spin(V,g)$ is a representation of these groups. As the following theorem shows the restrictions of $\widetilde{Ad}$ to $Pin(V,g)$ and $Spin(V,g)$ are twofold covering maps.

Theorem 2.3 (Thm. 2.10 in [LM89]). *Let V be a vector space with a symmetric nondegenerate bilinear form g. If the signature of g is different from $(1,1)$, $Pin(V,g)$ and $Spin(V,g)$ are two-sheeted coverings of $O(V,g)$ and $SO(V,g)$, respectively. Over each component of $O(V,g)$ the cover is nontrivial.*

In the following cases, which include 1 + 3-dimensional Minkowski space and the Lorentz group, the Spin groups (more precisely, their connected components of the identity) are the universal covering groups. Denote the identity components of $Spin(V,g)$ and $SO(V,g)$ by $Spin(V,g)_0$ and $SO(V,g)_0$, respectively, and recall that $SO(\mathbb{R}^{1+3},g)_0 = L_+^\uparrow$.

Corollary 2.10. *$Spin(V,g)_0$ is the universal cover of $SO(V,g)_0$, if the signature of g is $(1,r)$, $(r,1)$ or $(r,0)$ for $r \geq 3$.*

$\mathbb{R}^3$ or $\mathbb{R}^{1+3}$ are always considered as equipped with the Euclidean and Minkowski metric, respectively, so the metric is not explicitly indicated for the associated spin groups and we simply write $Cl(\mathbb{R}^{1+3})$, $Spin(\mathbb{R}^3)_0$ and $Spin(\mathbb{R}^{1+3})_0$. The cover $Spin(\mathbb{R}^3)$ of the rotation group $SO(\mathbb{R}^3)$ is simply connected, so that $Spin(\mathbb{R}^3)_0 = Spin(\mathbb{R}^3)$. This holds generally for Euclidean vector spaces of dimension greater than or equal to three.

3 An Example: Rotation Invariance

The example of a rotationally invariant quantum field theory is presented in this chapter. It is based on the construction given in [Kuc05]. The motivation for the approach, which will be generalised for the Lorentz group in the following chapter, can be described as follows. Applying the equation (2.29) characterising modular P_1CT symmetry for the field algebras twice leads to

$$J_a J_b \,\mathbf{F}(\mathcal{O})\, J_b J_a = \mathbf{F}(j_a j_b \mathcal{O}),$$

where, as usual, $a, b \in \mathcal{W}$ denote wedge regions and $\mathcal{O}$ is an open set in Minkowski space. Since $j_a j_b$ is an element of the orthogonal group $SO(\mathbb{R}^{1+3})_0 = L_+^\uparrow$, it is natural to ask whether it is possible to define a representation of the Lorentz group by simply setting $\tilde{W}(\Lambda) := J_a J_b$ for a pair of reflections j_a, j_b with $\Lambda = j_a j_b$. If this works, it could yield a covariant representation. In [BS04] this question was answered in the affirmative for modular conjugations associated with a bosonic net of algebras, for which $J_a = J_{-a}$ holds. In the case of general statistics one has to find a representation of the universal cover of the Lorentz group. Since $J_a \neq J_{-a}$, a characterisation of $\tilde{L}_+^\uparrow$ by pairs of wedges and not by pairs of reflections is required. The second problem is whether for the product of two unitaries $J_a J_b$ and $J_c J_d$ a pair of wedges e, f with

$$(J_a J_b)(J_c J_d) = J_e J_f$$

exists in such a way, that the map $\tilde{W}$ is a homomorphism. This leads to the question whether there is there a surjective map $\pi : \mathcal{W} \times \mathcal{W} \to \tilde{L}_+^\uparrow$ with the property

$$\pi(a, b)\pi(c, d) = \pi(e, f) \quad \Rightarrow \quad \begin{cases} (j_a j_b)(j_c j_d) & = j_e j_f \\ (J_a J_b)(J_c J_d) & = J_e J_f. \end{cases} \tag{3.1}$$

Pursuing this road further one arrives at a characterisation of the universal cover of the Lorentz group by pairs of wedges. Since the case of the rotation group is considerably more simple, we will restrict ourselves to this model in this chapter. It has been worked out in [Kuc05] and illustrates the program we want to follow for the more general case of Lorentz and finally Poincaré covariance. Some of the technical obstacles one encounters in the more complicated cases are already present here. The Lorentz- and Poincaré covariant problem will be treated in Chapters 4 and 5, respectively.

Starting point is the assumption of modular P_1CT-symmetry described in Section 2.2.3 and equation (2.28), motivating the construction of a model of the universal covering group of $SO(3)$ based on pairs of reflections. The example given in [Kuc05] will be referred to as $\mathbf{G}_R$. It will be introduced first, followed by an analysis of the Spin group $Spin(\mathbb{R}^3)_0$. The Spin group provides a more elegant way to relate the covering group to reflections in edges of wedges. Both models are isomorphic in a natural way and may be used to define a geometrically well-motivated, unitary and rotation covariant representation of the universal covering group of $SO(3)$ in terms of pairs of modular conjugations [Kuc05, Lor07]. For a quantum field obeying the assumptions stated in Section 2.3 together with this representation the spin-statistics theorem is straightforward to derive.

In this chapter only those pairs of wedges (and their associated modular conjugations) have to be considered, which generate a rotation. So fix a time direction by choosing a timelike unit vector e_0. Then any pair of wedges in

$$\mathcal{W}_{e_0} = \{W_\xi;\ \xi \in Z \text{ and } t_\xi = e_0\}$$

(cf. equation 2.5) generates a rotation. This will be elaborated in the following section.

3.1 $SO(3)$ and Reflections

It is a classical result of Cartan and Dieudonné (see e.g. [Art57]) that elements of the orthogonal groups in d dimensions can be decomposed into reflections. For any element one needs at most d reflections in the orthogonal complement of vectors, and for the special orthogonal group the required number of reflections is even. So we see that any rotation in three dimensions can be written as the product of two reflections in two-dimensional subspaces. In the following we will use the term *plane* to refer to a two-dimensional subspace of the corresponding vector space $\mathbb{R}^3$ or $\mathbb{R}^{1+3}$ (and not to an affine subspace).

For the rotation group in three dimensions there exists a simple geometric picture describing the possible decompositions of a rotation into two reflections in planes. In figure 3.1 a pair of vectors (a, b) is depicted with a projection of the planes of reflection indicated by the solid lines.

Performing the successive reflections in $a^\perp$ and $b^\perp$ yields a rotation with doubled angle compared to the angle between the reflection planes. The axis of rotation is the intersection of the two planes. Consider a nontrivial rotation ρ and a pair of vectors (a, b) generating it. Then every other pair of vectors generating the same rotation can be transformed into (a, b) or $(a, -b)$ by a rotation about the same axis. Note that for a pair (a, b) generating a nontrivial rotation a and b are always perpendicular to the axis of the rotation.

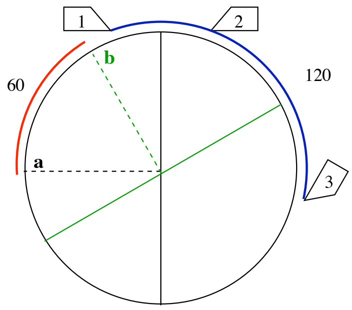


Figure 3.1: Decomposition of a rotation into two reflections. Object 1 is mapped to object 2 by reflection in $a^\perp$. Reflection in $b^\perp$ maps object 2 to object 3. Together they yield a rotation by twice the angle between a and b.

For any zweibein ξ characterising a wedge in $\mathcal{W}_{e_0}$ the timelike vector $t_\xi = e_0$ is fixed. The spacelike unit vector x_ξ is a time-zero vector and therefore an element of the three-dimensional subspace $e_0^\perp$ of $\mathbb{R}^{1+3}$. For two wedges $W_\xi, W_\chi \in W_{e_0}$ one has

$$j_\xi j_\chi = j_{(e_0,x_\xi)} j_{(e_0,x_\chi)} = j_{e_0} j_{x_\xi} j_{e_0} j_{x_\chi} = j_{x_\xi} j_{e_0} j_{e_0} j_{x_\chi} = j_{x_\xi} j_{x_\chi},$$

which is a rotation about an axis orthogonal to e_0, x_ξ and x_χ. The wedges in $\mathcal{W}_{e_0}$ and the products of the associated reflections are therefore determined by spacelike unit vectors in $e_0^\perp$.

If one wants to define a representation of the universal covering group of $SO(3)$ in terms of pairs of modular conjugations as described above, then it suffices to characterise the universal covering group by pairs of unit vectors in $e_0^\perp$, which is isomorphic with $\mathbb{R}^3$. In the following section we will identify $e_0^\perp$ with $\mathbb{R}^3$. Every vector in $\mathbb{R}^3$ specifies a wedge in $\mathcal{W}_{e_0}$ and an associated modular conjugation.

3.2 Reflections and the Universal Cover of $SO(3)$

One knows that the universal cover of the rotation group in three dimensions (for example given by $SU(2)$) is two-sheeted. The fact that one can describe a plane by a normal vector for which one has two choices motivates the construction of a universal covering group in terms of pairs of vectors and their associated reflections.

3.2.1 The Group G_R

The characterisation of the universal cover of the rotation group by pairs of wedges leads to the group $\mathbf{G}_R$ defined in this section. The map π announced in the introduction to this chapter will be played by a projection map associated with an appropriate equivalence relation. Consider pairs of unit vectors in $\mathbb{R}^3$ (which is identified with the subspace $e_0^\perp$ of $\mathbb{R}^{1+3}$) and introduce the following equivalence relation.

Definition 3.1. *All pairs of vectors of the form (a,a) are equivalent and the same holds for all pairs of the form $(a,-a)$. Two pairs of unit vectors (a,b) and (c,d) which do not fall into these cases generate a nontrivial rotation. They are called equivalent if there exists a rotation $\rho \in SO(3)$ about the same axis as $j_a j_b$ with $(\rho a, \rho b) = (c,d)$. This implies $j_a j_b = j_c j_d$ and $\rho j_a j_b = j_a j_b \rho$. The canonical projection map is denoted by π.*

So for two equivalent pairs of vectors generating a nontrivial rotation one pair can be rotated into the other by a rotation about the same axis. Figure 3.2 illustrates this. The solid lines represent the planes of reflection.

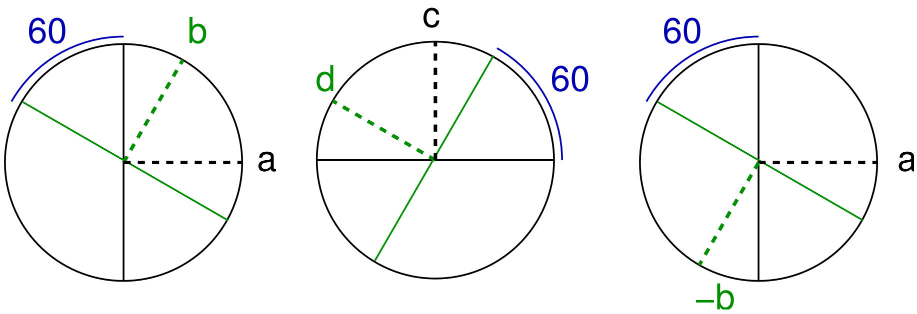


Figure 3.2: Illustration of the equivalence relation. The pairs of vectors (a,b), (c,d) and $(a,-b)$ generate the same rotation with angle 120 degrees. (a,b) is equivalent to (c,d), but not to $(a,-b)$.

The equivalence classes $\pi(a,b)$ have the following properties. A map $\tilde{\lambda}$ mapping $\pi(a,b)$ to the rotation $j_a j_b$ is well-defined, and one can write $a = \tau b$, where τ is a rotation about the same axis as $j_a j_b$ and $\tau^2 = j_a j_b$. If $j_a j_b$ is trivial so that the rotation axis is not defined, then $\tau^2 = 1$ and $\tau a = \pm b$. One can verify that the fibre of $\tilde{\lambda}$ over $j_a j_b$ contains precisely two elements, namely $\pi(a,b)$ and $\pi(a,-b)$.

In [Kuc05, Section 2] it was proven that $\mathbf{G}_R := S^2 \times S^2/\sim$ is indeed a realisation of the universal cover of $SO(3)$. Let us briefly indicate the ideas of the proof and point out the technical difficulties one has to overcome. Let λ be the map

$$\lambda : S^2 \times S^2 \to SO(3); \quad (a,b) \mapsto j_a j_b.$$

The goal is to show that $\mathbf{G}_R$ is a simply connected Hausdorff space and that there is a covering map $\tilde{\lambda} : \mathbf{G}_R \to SO(3)$ such that the diagram

$$\begin{array}{ccc} S^2 \times S^2 & \xrightarrow{\pi} & \mathbf{G}_R \\ {\scriptstyle\lambda}\downarrow & \swarrow{\scriptstyle\tilde{\lambda}} & \\ SO(3) & & \end{array} \tag{3.2}$$

commutes. But, although the space $S^2 \times S^2$ is topologically well behaved, it is not clear from the outset that this also holds for $\mathbf{G}_R$. The problem arises because a factor space does not automatically inherit the topological properties of the original space. If π or λ were open, the Hausdorff property of $\mathbf{G}_R$ would be obvious. But in fact λ and π are only open on $\tilde{\lambda}^{-1}(SO(3) \setminus 1)$. The following lemma which is straightforward to prove, gives a tool at hand to check whether a map is open.

Lemma 3.2. *Let X, Y be first countable topological spaces and let $f : X \to Y$ be continuous. f is open if and only if one finds for every $y \in Y$ and sequence $(y_n)_n$ converging to y and for every $x \in f^{-1}(y)$ a sequence in X converging to x with $f(x_n) = y_n$ for all $n \in \mathbb{N}$.*

Proof. First one has to verify that a set M in a first countable space Z is open if and only if for every $x \in M$ and every sequence $(x_n)_n$ converging to x there exists $N \in \mathbb{N}$ with $x_n \in M$ for every $n \geq N$. Let the latter condition hold and assume M not to be open. There exists $x \in M$ so that there is no open neighbourhood of x contained in M. Since Z is first countable, there exists an open neighbourhood base $(U_\nu)_{\nu \in \mathbb{N}}$ of x, and $U_\nu \nsubseteq M$ for all $\nu \in \mathbb{N}$. For every ν choose $x_\nu \in U_\nu \setminus M$, then $(x_\nu)_\nu$ converges to x. But $x_\nu \notin M$ for all ν and this is a contradiction. The other direction follows by definition of convergence.

Now let $U \subset X$ be an open set in X. Then we want to show that $V := f(U)$ is open. Let $(y_n)_n$ be a sequence in Y, converging to $y \in V$ and let $x \in f^{-1}(V) \cap U$. By assumption there exists a sequence $(x_n)_n$ in X with $f(x_n) = y_n$ converging to x. Thus one finds $N \in \mathbb{N}$ with $x_n \in U$ for all $n > N$. Consequently $f(x_n) = y_n \in f(U)$ for all $n > N$, which proves one direction. Now assume f to be open. Let $y \in Y, x \in f^{-1}(y)$ and let $(U_\nu)_\nu$ be a neighbourhood basis of x. Setting $V_\nu := f(U_\nu)$ yields a neighbourhood basis of y. For every $k \in \mathbb{N}$ there exists $N_k \in \mathbb{N}$ such that $y_n \in V_k$ for all $n > N_k$, because $(y_n)_n$ converges to y. Construct a sequence $(x_n)_n$ by choosing $x_n \in f^{-1}(y_n)$ for $n \leq N_1$ and $x_n \in U_k \cap f^{-1}(y_n)$ for $N_k < n \leq N_{k+1}$. This sequence converges to x by construction and the proof is completed. □

With this lemma it is easy to verify that λ is not open on $\mathbf{G}_R$. Take a sequence of rotations $(\rho_n)_n$ in $SO(3)$ about a fixed axis $a \in S^2$ converging to 1. The element (a, a) is in $\lambda^{-1}(1)$. But since a is the axis of every ρ_n, for any $(b_n, c_n) \in \lambda^{-1}(\rho_n)$ one has that a is perpendicular to b_n and c_n. So it is impossible to find a sequence (b_n, c_n) with $\lambda(b_n, c_n) = \rho_n$ converging to (a, a). By the preceding lemma, λ cannot be open.

But one can establish the Hausdorff property relatively easy for $\dot{\mathbf{G}}_R := \mathbf{G}_R \setminus \lambda^{-1}(1)$ by constructing a homeomorphism to $B_1 \setminus \{0\}$, the dotted unit ball in $\mathbb{R}^3$. For $\pi(a, b)$ the homeomorphism is given by multiplying the vector $\dfrac{a \times b}{\pi |a \times b|}$ with the angle between a and b in the interval $(0, \pi)$. In the next step one singles out disjoint neighbourhoods for $g \in \tilde{\lambda}^{-1}(1)$ and $h \in \mathbf{G}_R$, proving the Hausdorff property for all of $\mathbf{G}_R$. For details we refer to [Kuc05, Lemma 1].

Finally one proves that $\tilde{\lambda}$ is a local homeomorphism on $\dot{\mathbf{G}}_R$ and extends this to $\mathbf{G}_R$ with the help of Lemma 3.2. $\mathbf{G}_R$ is simply connected, because it is pathwise connected and the fundamental group of $SO(3)$ is isomorphic to $\mathbb{Z}_2$. The same strategy of proving, e.g. the Hausdorff property for an open subset of the quotient space and extending this to afterwards will also be followed for the more complicated case of the Lorentz group. As we have seen, the main obstacles to overcome in the proof of $\mathbf{G}_R$ being a universal covering space are of a topological nature. The geometrical ideas underlying the construction of $\mathbf{G}_R$ and the covering map $\tilde{\lambda}$ are simple and intuitive.

3.2.2 Isomorphism of $Spin(\mathbb{R}^3)_0$ and $\mathbf{G}_R$

Now we want to describe the spin group $Spin(\mathbb{R}^3)_0$ by pairs of unit vectors in $\mathbb{R}^3$. To clarify the statements we will explicitly take care of the injection of $S^2 \subset \mathbb{R}^3$ in $Spin(\mathbb{R}^3)_0$ and use the notation $\underline{a}$ to distinguish $a \in S^2$ from the embedded element $\underline{a} \in Spin(\mathbb{R}^3)_0$. Consequently, the embedding of S^2 in $Cl(\mathbb{R}^3)$ is denoted by $\underline{S}^2$. This is more convenient than employing the embedding $\iota : \mathbb{R}^3 \to Cl(\mathbb{R}^3)$ defined in Section 2.4.1. In subsequent chapters the distinction will be completely neglected.

The following lemma shows that the spin group $Spin(\mathbb{R}^3)_0$, which by definition is generated by products of unit vectors with an even number of factors, can in fact be described by pairs of vectors.

Lemma 3.3. *Any element of $Spin(\mathbb{R}^3)_0$ can be written as the product of two unit vectors, justifying the notation*

$$Spin(\mathbb{R}^3)_0 = \{\underline{a} \cdot \underline{b};\ a, b \in S^2\} =: \underline{S}^2 \cdot \underline{S}^2.$$

Proof. Consider the two-sheeted covering map

$$\tilde{\lambda}_R : Spin(\mathbb{R}^3)_0 \to SO(3); \quad g \mapsto \tilde{\lambda}_R(g) := \widetilde{Ad}_g.$$

$\tilde{\lambda}_R|_{\underline{S}^2 \cdot \underline{S}^2}$ is surjective because every rotation can be written as the product of two reflections $j_a j_b$, so $\underline{a} \cdot \underline{b} \in \tilde{\lambda}_R^{-1}(j_a j_b) \cap \underline{S}^2 \cdot \underline{S}^2$. Every fibre over $\tilde{\lambda}(\underline{a} \cdot \underline{b})$ contains the two elements $\pm \underline{a} \cdot \underline{b} \in \underline{S}^2 \cdot \underline{S}^2$. So the subset $\underline{S}^2 \cdot \underline{S}^2$ of $Spin(\mathbb{R}^3)_0$ is a universal covering space of $SO(3)$, and consequently $\underline{S}^2 \cdot \underline{S}^2$ and $Spin(\mathbb{R}^3)_0$ have to coincide. □

As one can see, the description of $Spin(\mathbb{R}^3)_0$ in terms of pairs of reflections is straightforward and notably more elegant than the realisation $\mathbf{G}_R$. Since $\mathbf{G}_R$ and $Spin(\mathbb{R}^3)_0$ are models for the universal covering group of $SO(3)$, they have to be isomorphic. In both realisations the elements can be characterised by pairs of reflections, indicating that an explicit isomorphism may be given in a simple way. The following lemmas serve as a preparation to the definition of the isomorphism $\mathbf{G}_R \cong Spin(\mathbb{R}^3)_0$. For $a, b \in S^2$ the product $\underline{a} \cdot \underline{b}$ in $Cl(\mathbb{R}^3)$ is abbreviated by $\underline{ab}$.

Lemma 3.4. *For wedges $a, b, c, d \in S^2$ one has $\underline{aba} = \underline{-j_a b}$. This implies $\underline{ab}\,\underline{cd}\,\underline{ba} = \underline{(j_a j_b c)}\underline{(j_a j_b d)}$.*

Proof. $\tilde{\lambda}(\underline{aba}) = j_a j_b j_a = j_{j_a b}$ implies $\underline{aba} = \pm \underline{j_a b}$. The map Γ_a given by $\Gamma_a(b) = \underline{aba}(\underline{-j_a b})$ is continuous, and $\Gamma_a(a) = \mathbb{1}$. Since $\tilde{\lambda}(\Gamma_a(b)) = 1$ for all $a, b \in S^2$, this implies $\Gamma_a(b) = \mathbb{1}$ by continuity of Γ_a. The second statement follows immediately from $\underline{ab}\,\underline{cd}\,\underline{ba} = \underline{ab}\,\underline{c}\,\underline{ba}\,\underline{ab}\,\underline{d}\,\underline{ba}$. □

For any group G define the commutator by $[g, h] := ghg^{-1}h^{-1}$ for $g, h \in G$. For the covering group $Spin(\mathbb{R}^3)_0$ the nontrivial element in the fibre over the identity element of $SO(3)$ is denoted by $-\mathbb{1}$.

Lemma 3.5. *Let $[\tilde{\lambda}(g), \tilde{\lambda}(h)] = 1$ for $g, h \in \underline{S}^2 \cdot \underline{S}^2$. Then $[g, h] = \pm\mathbb{1}$ and $[g^2, h] = \mathbb{1}$.*

Proof. By direct computation one shows $\tilde{\lambda}([g, h]) = 1$, then the statements follow immediately. □

In the following case the condition $[\tilde{\lambda}(g), \tilde{\lambda}(h)] = 1$ is sufficient for g and h to commute. If there exists $\rho \in SO(3)$ with $\rho^2 = \tilde{\lambda}(g)$ and $[\tilde{\lambda}(g), \rho] = 1$, then $[g, h] = 1$: Choose $g_1 \in \tilde{\lambda}(\rho)^{-1}$. Then one has $g_1^2 = \pm g$ for one of the signs and $gh = \pm g_1^2 h = \pm h g_1^2 = hg$.

Consider two unequal pairs of unit vectors, (a, b) and (c, d). If there exists a (nontrivial) rotation σ about the same axis as $j_a j_b$ with $\sigma(a, b) = (c, d)$, then let τ be another rotation about the same axis with $\tau^2 = \sigma$. Choosing $h \in \tilde{\lambda}^{-1}(\tau)$, Lemma 3.5 implies

$$\underline{c}\,\underline{d} = (\underline{\sigma a})(\underline{\sigma b}) = h^2 \underline{a}\,\underline{b} h^{-2} = \underline{a}\,\underline{b}. \tag{3.3}$$

Conversely, if $\underline{ab} = \underline{cd}$, then the rotations $j_a j_b$ and $j_c j_d$ coincide. As remarked at the end of Section 3.1, this implies the existence of a rotation σ about the same axis as $j_a j_b$ with $\sigma(a, b) = (c, \pm d)$. Assuming $\sigma(a, b) = (c, -d)$ leads to the contradiction $\underline{ab} = -\underline{cd} = \underline{cd}$ by applying equation (3.3). This proves the isomorphism $Spin(\mathbb{R}^3)_0 \cong \mathbf{G}_R$, since $\underline{ab} = \underline{cd}$ if and only if $(a, b) \sim (c, d)$. We formulate this as

Theorem 3.1. *The map ι_R given by*

$$\iota_R : \mathbf{G}_R \to Spin(\mathbb{R}^3)_0 = \underline{S}^2 \cdot \underline{S}^2; \quad g = \pi(a, b) \mapsto \underline{ab} \tag{3.4}$$

is well-defined and an isomorphism.

3.3 Construction of the Representation and the Spin-Statistics Theorem

The goal is to define a unitary representation $\tilde{W}$ of the universal covering group of $SO(3)$. This representation should act covariantly on the field, i.e.

$$\tilde{W}(g) F(\mathfrak{c}, f) \tilde{W}(g)^\dagger = F(\tilde{D}(g)\mathfrak{c}, f(\tilde{\lambda}_R(g) \cdot\,)) \tag{3.5}$$

where $\tilde{D}$ is a representation in the component space $\mathfrak{C}$. The quantum field treated in this section obeys the generalised Wightman axioms given on page 33 in Section 2.3.3. A timelike unit vector $e_0 \in \mathbb{R}^{1+3}$ is fixed to specify a time direction and render the notion of rotations in Minkowski space meaningful. A rotation is a Lorentz transformation leaving e_0 and a spacelike vector invariant.

In the last section the universal covering group of $SO(3)$ was characterised by pairs of wedges from $\mathcal{W}_{e_0}$. Elements of $\mathcal{W}_{e_0}$ are determined by unit vectors in the three-dimensional subspace $e_0^\perp$ of Minkowski space. This subspace is identified with $\mathbb{R}^3$. For a unit vector $a \in e_0^\perp \cong \mathbb{R}^3$ the modular objects and field algebras associated with the wedge $W_{(e_0,a)} \in \mathcal{W}_{e_0}$ will be denoted by J_a, Δ_a and $\mathbf{F}(a)$, respectively. Recall that the self-adjoint and unitary statistics operator k satisfies $k\mathbf{F}(a)k = \mathbf{F}(a)$. By the assumption of modular $\mathrm{P_1CT}$ symmetry, the field algebras satisfy $J_a\mathbf{F}(b)J_a = \kappa\mathbf{F}(j_a b)\kappa^\dagger =: \mathbf{F}^t(j_a b)$. κ was defined as $\kappa = (1+ik)/(1-ik)$. Some important relations for the modular objects associated with wedges follow from Proposition 2.6 on page 26 for modular objects in algebras of unbounded operators. These are not restricted to the rotation invariant case and will be also needed in the subsequent chapters.

Lemma 3.6. *Let U be a unitary or antiunitary operator with*

$$U\mathbf{F}(a)U^\dagger = \mathbf{F}^t(b)$$

for $a \in \mathcal{W}$ or $a \in S^2$, then

$$U\Delta_a^{1/2}U^\dagger = \Delta_b^{1/2} \quad \text{and} \quad UJ_aU^\dagger = J_{-b}.$$

Proof. First note that $J_a\mathbf{F}(a)J_a = \mathbf{F}^t(-a)$, and therefore $\kappa^\dagger J_a\mathbf{F}(a)J_a\kappa = \mathbf{F}(-a)$. Proposition 2.6 implies $J_{-a} = \kappa J_a J_a J_a \kappa^\dagger = \kappa J_a \kappa^\dagger$ and therefore $\kappa^\dagger U J_a U^\dagger \kappa = J_b$. The statement now follows because $\kappa^\dagger U \Delta_a^{1/2} U^\dagger \kappa = \Delta_b$ and κ commutes with Δ_b. □

Proposition 2.6 and the preceding lemma yield a couple of important relations. For each $a \in \mathcal{W}$ one has $k\mathbf{F}(a)k^\dagger = k\mathbf{F}(a)k = \mathbf{F}(a)$, so

$$kJ_ak = J_a, \quad \text{whence} \quad J_a\kappa = \kappa^\dagger J_a \tag{3.6}$$

follows by antilinearity of J_a. By modular $\mathrm{P_1CT}$ symmetry, $J_a\mathbf{F}(a)J_a = \mathbf{F}^t(-a)$, so

$$\kappa^\dagger J_a \kappa = \kappa^\dagger J_a J_a J_a \kappa = J_{-a}. \tag{3.7}$$

It also follows from modular $\mathrm{P_1CT}$ symmetry that $J_a\mathbf{F}(b)J_a = \mathbf{F}^t(j_a b)$, so

$$J_a J_b J_a = \kappa J_{j_a b} \kappa^\dagger = J_{-j_a b}. \tag{3.8}$$

Note that the minus sign in the last equality is important, in contrast to the case of reflections where $j_a j_b j_a = j_{\pm j_a b}$ (cf. eq. (2.32)). By the same reasoning as for the action on modular conjugations one finds for the action on the modular operators that $J_a \Delta_b J_a = \Delta_{j_a b}$. Now we can define the representation. In the proof the assumption of continuity of the map $a \mapsto J_a$ enters.

Theorem 3.2 (Thm. 5.(iii) in [Kuc05]). *Identify* S^2 *with the unit vectors in* $e_0^\perp \subset \mathbb{R}^{1+3}$. *For* $g \in Spin(\mathbb{R}^3)_0$ *take* $a, b \in S^2$ *with* $g = \underline{ab}$ *and set* $\tilde{W}(\underline{ab}) := J_{(e_0,a)} J_{(e_0,b)}$. *The following statements hold.*

(i) $\tilde{W}$ *is well-defined.*

(ii) $\tilde{W}$ *is a homomorphism.*

Proof. The proof of (i) is based on the proof of [BS04, Lemma 2.4]. We will give a simplified version of the proof of (ii) (cf. [KL07]) taking advantage of the simpler algebraic properties of $Spin(\mathbb{R}^3)_0$ compared to $\mathbf{G}_R$. Let $g, h \in Spin(\mathbb{R}^3)_0$ and choose $a, b, c \in S^2$ with $g = \underline{ab}$ and $h = \underline{bc}$. This is possible because the planes orthogonal to the axes of the rotations $\tilde{\lambda}(g)$ and $\tilde{\lambda}(h)$ have nonempty intersection. Then $\tilde{W}(g)\tilde{W}(h) = \tilde{W}(\underline{ab})\tilde{W}(\underline{bc}) = J_a J_b J_b J_c = J_a J_c = \tilde{W}(\underline{ac}) = \tilde{W}(\underline{abbc}) = \tilde{W}(gh)$. □

Using modular $\mathrm{P_1CT}$ symmetry once more, a representation of $Spin(\mathbb{R}^3)_0$ in the component space will be defined. Here the technical assumption for the component space to be "free of redundancies" enters (see assumptions on page 33 f.).

Theorem 3.3. *There exists a representation* $\tilde{D}$ *of* $Spin(\mathbb{R}^3)_0$ *in* $\mathfrak{C}$ *so that* $\tilde{W}$ *acts covariantly with respect to rotations (cf. eq. (3.5)).*

Proof. Define a map D from $S^2 \times S^2$ into the automorphism group $Aut(\mathfrak{C})$ of $\mathfrak{C}$ by $D(a, b) := C_a C_b$. If $\underline{ab} = \underline{cd}$, then modular $\mathrm{P_1CT}$ symmetry and Theorem 3.2 imply

$$\begin{aligned} F(C_a C_b \mathfrak{c}, j_a j_b f) &= \tilde{W}(\underline{ab}) F(\mathfrak{c}, f) \tilde{W}(\underline{ab})^\dagger \\ &= \tilde{W}(\underline{cd}) F(\mathfrak{c}, f) \tilde{W}(\underline{cd})^\dagger \\ &= F(C_c C_d \mathfrak{c}, f(j_c j_d(\,\cdot\,))) \\ &= F(C_c C_d \mathfrak{c}, f(j_a j_b(\,\cdot\,))) \end{aligned}$$

for all $\mathfrak{c}$ and all f. Using the assumption concerning redundancies in the component space (page 33f.), one obtains $C_a C_b \mathfrak{c} = C_c C_d \mathfrak{c}$ for all $\mathfrak{c}$, so $D(a, b) = D(c, d)$; and a map $\tilde{D} : Spin(\mathbb{R}^3)_0 \to Aut(\mathfrak{C})$ may be defined by $\tilde{D}(\underline{ab})) := D(a, b)$. This map $\tilde{D}$ now inherits the representation property from $\tilde{W}$. □

Now the ground is prepared for the spin-statistics theorem.

Theorem 3.4 (Spin-statistics theorem). *The representation* $\tilde{W}$ *of the universal cover of the Lorentz group satisfies*

$$\tilde{W}(-\mathbb{1}) = k.$$

For all $\mathfrak{c} \in \mathfrak{C}$ *and all* $f \in \mathcal{S}$ *one has*

$$F_\pm(\mathfrak{c}, f) = \frac{1}{2}\Big(F(\mathfrak{c}, f) \pm F(\tilde{D}(-\mathbb{1})\mathfrak{c}, f)\Big)$$

for the bosonic and fermionic components of the field. If $\tilde{D}$ *is irreducible with spin* s*, then* $\tilde{D}(-\mathbb{1}) = \exp(2\pi i s)$*, and therefore* $F_- = 0$ *for bosonic fields and* $F_+ = 0$ *for fermionic fields.*

Proof. For each $a \in S^2$ one calculates

$$\tilde{W}(-\mathbb{1}) = J_a J_{-a} = J_a \kappa J_a \kappa^\dagger = J_a^2 (\kappa^\dagger)^2 = k,$$

so

$$\begin{aligned} kF(\mathfrak{c}, f)k &= \tilde{W}(-\mathbb{1}) F(\mathfrak{c}, f) \tilde{W}(-\mathbb{1}) \\ &= \tilde{W}(-\mathbb{1}) F(\mathfrak{c}, f) \tilde{W}(-\mathbb{1})^\dagger = F(\tilde{D}(-\mathbb{1})\mathfrak{c}, f). \end{aligned}$$

□

This is the well-known spin-statistics theorem in quantum field theory. Bosonic fields cannot obey anticommutation relations and fermionic fields cannot obey commutation relations. The strategy followed successfully for this example of a rotation-covariant quantum field theory will be applied to a Lorentz invariant theory in the next chapter.

4 Lorentz Invariance

Following the ideas illustrated in Chapter 3 for the example of a rotationally invariant quantum field theory, the analysis will now be extended to the case of a Lorentz invariant theory. Most of the results presented in this chapter have been published in [KL07]. The representation to be introduced will again be based on pairs of modular conjugations. By assumption the adjoint action of a pair of modular conjugations on a field operator induces a Lorentz transformation given by a product of two reflections in edges of wedges. This will be elaborated in the following section. As in the preceding chapter two characterisations of the universal covering group by pairs of wedges will be introduced. The realisation $\mathbf{G}_L$ is, similar as $\mathbf{G}_R$, defined as the quotient space of a suitable equivalence relation on pairs of wedges. The covering map is simply defined by taking two wedges in an equivalence class and calculating the product of the two reflections in their edges. As for the rotation group, the tedious part consists in checking the topological properties of the quotient space (Section 4.2.1). But the geometrical results are also useful for the analysis of the spin group $Spin(\mathbb{R}^{1+3})_0$, defined as a subgroup of the Clifford algebra associated with Minkowski space. The spin group $Spin(\mathbb{R}^{1+3})_0$ provides the starting point for the second characterisation of $\tilde{L}_+^\uparrow$ by pairs of wedges. It is a two-sheeted cover of the Lorentz group, and the covering map is closely related with reflections in hyperplanes. This realisation of the universal cover of the Lorentz group and its characterisation seems to be more elegant than the construction of $\mathbf{G}_L$ and is the topic of [Lor07]. The identification of $Spin(\mathbb{R}^{1+3})_0$ with pairs of wedges will be derived in Section 4.2.3. The geometrical structure encoded in both realisations of $\tilde{L}_+^\uparrow$ is essential for constructing the representation of $\tilde{L}_+^\uparrow$. They allow for establishing the independence of the product of modular conjugations $J_a J_b$ of the pair of wedges specifying an element of the covering group of $L_+^\uparrow$. Furthermore, the properties of $\mathbf{G}_L$ and $Spin(\mathbb{R}^{1+3})_0$ enter in the proof of the fact that a product of four modular conjugations can be reduced to a product of two modular conjugations.

The derivation of the spin-statistics theorem then proceeds exactly as before. It will be convenient to shorthand $L_+^\uparrow$, the connected component of the identity of L, by L_0 in this chapter.

4.1 Lorentz Group and Reflections

We mentioned before the well-known theorem of Cartan-Dieudonné stating that any element of an orthogonal group associated with a symmetric bilinear form can be decomposed into reflections in hyperplanes. But unlike in the case of rotations it is not a trivial fact that elements of the Lorentz group can be decomposed into two reflections in two-dimensional spacelike subspaces (edges of wedges). This has been shown in [BS04] and, in a broader context, in [Ell04]. Since the geometrical arguments in [BS04] are important for the understanding of the subsequent discussion of the models $\mathbf{G}_L$ and $Spin(\mathbb{R}^{1+3})_0$, we will recall them here. Because they rely on the rotation-boost decomposition in L_0, we have to specify a time direction by fixing a timelike unit vector e_0. Recall that, by definition, a rotation leaves e_0 and a spacelike vector invariant, and a boost leaves a two-dimensional spacelike subspace of the hyperplane $e_0^\perp$ pointwise invariant. A convenient and coordinate-independent description of the set of wedges in Minkowski space is given by specific pairs of vectors. The set of *zweibeine*

$$Z = \{\xi := (t_\xi, x_\xi) \in \mathbb{R}^{1+3} \times \mathbb{R}^{1+3};\ g(t_\xi, t_\xi) = 1, g(x_\xi, x_\xi) = -1, g(t_\xi, x_\xi) = 0\} \quad (4.1)$$

was introduced in equation (2.3). Recall that any zweibein $\xi = (t_\xi, x_\xi)$ defines a wedge W_ξ via

$$W_\xi := \{x \in \mathbb{R}^{1+3}; -g(x, x_\xi) > |g(x, t_\xi)|\},$$

and that the set of all wedges which are Lorentz transforms of the right wedge was denoted by $\mathcal{W}$. To decompose a nontrivial rotation ρ into a product of two reflections in edges of wedges take a spacelike unit vector e orthogonal to the subspace spanned by the rotation axis and e_0. This subspace will be called the rotation plane. Now take a rotation τ about the same axis as ρ, whose square is ρ. Denote the wedge $W_{(e_0,e)}$ by $\bar{e}$. Then the following relations hold,

$$\rho W_{(e_0,e)} = W_{(\tau e_0, \tau e)} = W_{(e_0, \tau e)} \quad \text{and} \quad \rho = \rho j_{\bar{e}} j_{\bar{e}} = j_{\tau\bar{e}} j_{\bar{e}}.$$

A boost β can be written in a similar way. Take a spacelike unit vector e perpendicular to the boost direction of β and use again the shortcut $\bar{e} := W_{(e_0,e)}$. Then

$$\beta W_{(e_0,e)} = W_{(\beta^{1/2} e_0, \beta^{1/2} e)} = W_{(\beta^{1/2} e_0, e)} \quad \text{and} \quad \beta = j_{\bar{e}} j_{\bar{e}} \beta = j_{\bar{e}} j_{\beta^{-1/2}\bar{e}}.$$

For a general Lorentz transformation $\mu \in L_0$ the rotation-boost decomposition $\mu = \rho\beta$ is unique. There exists a spacelike unit vector e orthogonal to e_0, to the rotation axis of ρ and to the boost direction of β (if the rotation and boost are nontrivial). As above this yields for τ and $\bar{e} := W_{(e_0,e)}$

$$\mu = \rho\beta = j_{\tau\bar{e}} j_{\bar{e}} j_{\bar{e}} j_{\beta^{-1/2}\bar{e}} = j_{\tau\bar{e}} j_{\beta^{-1/2}\bar{e}}. \quad (4.2)$$

So every Lorentz transformation can be written as the product of two reflections in edges of wedges. Now we will derive some general properties of the Lorentz group and the representation of Lorentz transformations by reflections in edges of wedges needed in Sections 4.2.1 and 4.3.

Reflections of spacelike planes in spacelike planes It may be that the following lemma, which is highly plausible at a first glance, but somewhat tricky to prove, has been established earlier by other authors. But since such a reference is not known to us, we prove it here.

Proposition 4.1. *If A and B are two-dimensional spacelike subspaces of $\mathbb{R}^{1+3}$, then there exists a two-dimensional* spacelike *subspace C such that B is the image of A under orthogonal reflection by C.*

Proof. If A and B have nontrivial intersection, then there exist linearly independent nonzero vectors $a \in A$, $b \in B$, and $c \in A \cap B$. The one-dimensional timelike space $\{a, b, c\}^\perp$ is perpendicular to both A and B, so A and B are subspaces of a common time-zero plane, and the problem reduces to the well-known three-dimensional Euclidean case.

It remains to consider the case of A and B having trivial intersection. $A^\perp$ and $B^\perp$ are timelike planes and, hence, are spanned by future-directed lightlike vectors $x, y \in A^\perp$ and $v, w \in B^\perp$. Since A and B have trivial intersection, x, y, v, and w are linearly independent, so the inner product between any two distinct vectors of these is strictly positive.

Let C be the plane spanned by the vectors $x - \alpha v$ and $y - \beta w$, where

$$\alpha := \sqrt{\frac{g(x,y)}{g(v,w)}\frac{g(x,w)}{g(y,v)}} > 0 \qquad \text{and} \qquad \beta := \sqrt{\frac{g(x,y)}{g(v,w)}\frac{g(y,v)}{g(x,w)}} > 0.$$

We claim that $C^\perp$ is spanned by $x + \alpha v$ and $y + \beta w$. To prove this one shows that the vectors spanning C are orthogonal to the vectors spanning $C^\perp$. First calculate

$$g(x - \alpha v, x + \alpha v) = g(x,x) - \alpha^2 g(v,v) = 0 = g(y - \beta w, y + \beta w).$$

To check the other cases, note that $\alpha\beta = \dfrac{g(x,y)}{g(v,w)}$ and calculate

$$\begin{aligned} g(x - \alpha v, y + \beta w) &= g(x,y) - \alpha\beta\, g(v,w) - \alpha\, g(y,v) + \beta\, g(x,w) \\ &= g(x,y) - \frac{g(x,y)}{g(v,w)}\, g(v,w) \\ &\quad - \sqrt{\frac{g(x,y)g(x,w)}{g(v,w)g(y,v)}}\, g(y,v) + \sqrt{\frac{g(x,y)g(y,v)}{g(v,w)g(x,w)}}\, g(x,w) \\ &= -\sqrt{\frac{g(x,y)}{g(v,w)} g(x,w)g(y,v)} + \sqrt{\frac{g(x,y)}{g(v,w)} g(y,v)g(x,w)} \\ &= 0, \end{aligned}$$

and similarly $g(x + \alpha v, y - \beta w) = 0$.

$C^\perp$ is timelike because $g(x+\alpha v, x+\alpha v) = 2\alpha\, g(x,v) > 0$, so C is spacelike. Denote by j_C the orthogonal reflection by C. One then finds

$$j_C x = \tfrac{1}{2}\, j_C\big(\underbrace{(x+\alpha v)}_{\in C^\perp} + \underbrace{(x-\alpha v)}_{\in C}\big) = \tfrac{1}{2}\left(-(x+\alpha v) + (x-\alpha v)\right) = -\alpha v \in B^\perp$$

and $j_C y = -\beta w \in B^\perp$. □

In the rotation group, a pair of vectors spanning an angle α generates a rotation by an angle 2α (see figure 3.1). The following corollary provides an analogue to this observation for the Lorentz group.

Corollary 4.2. *Let $(a,b) \in \mathcal{W}\times\mathcal{W}$. There exists an element $c \in \mathcal{W}$ with $a = \pm j_c b$. With the abbreviation $j_c j_b =: \mu$, one has*

$$\mu^2 = j_a j_b = j_{\mu b} j_b$$

and therefore $a = \pm\mu b$.

Proof. Existence of c follows from Proposition 4.1. The other statements follow from

$$j_c j_b j_c j_b = j_{j_c b} j_b = j_a j_b \qquad \text{and} \qquad a = \pm j_c b = \mp j_c j_b b = \mp \mu b.$$ □

Square roots of Lorentz transformations and commutants In the following, some properties of square roots of Lorentz transformations and commutants of Lorentz transformations are established. It is likely that these are known, but the author is not aware of any reference. Some of the calculations are conveniently carried out in $SL(2,\mathbb{C})$, the universal cover of the Lorentz group. The results are needed for the realisation of the universal cover of $L_+^\uparrow$ in terms of wedges in Sections 4.2.1 and 4.2.3.

Lemma 4.3. *Let μ and ν be restricted Lorentz transformations.*

(i) Suppose that $\mu \neq 1$. There exists at least one and at most two elements ν with $\mu = \nu^2$. If, in particular, $\mu^2 = 1$, there are two square roots which are inverses of each other.

(ii) The commutant of μ is an abelian group if and only if $\mu^2 \neq 1$.

(iii) Given $\mu, \nu \in L_0$, suppose that $\mu^2 \neq 1 \neq \nu^2$ and $\mu^2\nu^2 = \nu^2\mu^2$. Then $\mu\nu = \nu\mu$.

Proof. The matrix group $SL(2,\mathbb{C})$ is known to be isomorphic to the universal covering group of L_0. Let Λ be any covering map from $SL(2,\mathbb{C})$ onto L_0. Then $\Lambda^{-1}(\Lambda(A)) = \pm A$ for any $A \in SL(2,\mathbb{C})$.

The conjugacy classes of $SL(2,\mathbb{C})$ are classified by the Jordan matrices in $SL(2,\mathbb{C})$,

$$N_z := \begin{pmatrix} z & 0 \\ 0 & 1/z \end{pmatrix}, z \in \dot{\mathbb{C}} \qquad N_\infty := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad N_{-\infty} := \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix},$$

so for each $A \in SL(2,\mathbb{C})$ there exists a $z \in \dot{\mathbb{C}} \cup \pm\infty$ and a $P \in SL(2,\mathbb{C})$ with $A = PN_zP^{-1}$.

Proof of (i). Since $\mu \neq 1$ by assumption and since $[N_{-z}] = [-N_z]$, there exists an element $A = PN_zP^{-1} \in \Lambda^{-1}(\mu)$ with $\pm 1 \neq z \neq -\infty$.

If $z \neq \infty$, the elements of $\Lambda^{-1}(\mu)$ are $\pm A$, and $B_\pm \in SL(2,\mathbb{C})$ satisfy $B_\pm^2 = \pm A$ if and only if $\pm B_\pm = \pm PN_{w_\pm}P^{-1}$ for complex square roots $w_\pm$ of $\pm z$. One obtains two square roots $\nu_\pm := \Lambda(B_\pm) \equiv \Lambda(-B_\pm)$ of μ.

If $z = \infty$, then $B_\pm := \pm P \begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix} P^{-1}$ are the two square roots of A. Since the elements in $[N_{-\infty}]$ have no square roots in $SL(2,\mathbb{C})$, the only square root of μ is $\nu := \Lambda(B_\pm)$.

If $\mu^2 = 1$ and $\nu^2 = \mu$, then $\nu^{-2} = \mu^{-1} = \mu$. The roots ν and ν^{-1} are distinct, since $\nu = \nu^{-1}$ would imply $1 = \nu\nu^{-1} = \nu^2 = \mu$, contradicting the assumption.

Proof of (ii). $\mu\nu = \nu\mu$ if and only if $AB = \pm BA$ for all $A \in \Lambda^{-1}(\mu)$ and $B \in \Lambda^{-1}(\nu)$.

Given $A = PN_zP^{-1} \in SL(2,\mathbb{C})$ with $z \neq \pm 1$, the commutant of A is the abelian group $\{PN_zP^{-1} : z \in \dot{\mathbb{C}}\}$.

The anticommutant of A is trivial if $z \neq \pm i$; otherwise it consists of the matrices $P \begin{pmatrix} 0 & v \\ -1/v & 0 \end{pmatrix} P^{-1}$. These matrices neither commute nor anticommute with the elements of the commutant of $PN_{\pm i}P^{-1}$.

But if $\mu^2 \neq 1$, then there exists an $A = PN_zP^{-1} \in \Lambda^{-1}(\mu)$ with $\pm 1 \neq z \neq \pm i$, so the commutant A^c of A is an abelian subgroup of $SL(2,\mathbb{C})$, and the anticommutant of A is trivial. Accordingly, the commutant μ^c of μ is the abelian group $\Lambda(A^c)$.

If $\mu^2 = 1$, all $z \in \mathbb{C}$ with $A = PN_zP^{-1}$ and $\Lambda(A) = \mu$ equal ± 1 or $\pm i$. If $z = \pm 1$, then $\mu = 1$, the commutant is L_0 and, hence, non-abelian, and if $z = \pm i$, the above remarks apply.

Proof of (iii). Since $\mu^2 \neq 1 \neq \nu^2$by assumption, it follows from the preceding statement that the commutants μ^c and ν^c are the maximal abelian groups $\{PN_zP^{-1} : z \in \dot{\mathbb{C}}\}$, $\{PN_zP^{-1} : z \in \dot{\mathbb{C}}\}$, or $\left\{P \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} P^{-1} : t \in \mathbb{C}\right\}$ for some $P \in SL(2,\mathbb{C})$. Consequently, the assumption $\nu^2 \in (\mu^2)^c$ implies $\nu \in (\mu^2)^c$, i.e., $\mu^2 \in (\nu^2)^c$. This yields the statement by the same argument. □

In the case of rotations it was easy to visualise that, for a pair of vectors (c,d) generating the same rotation ρ as a pair (a,b), one can find a rotation τ about the same axis as ρ with $(a,b) = \tau(c,d)$. In this case τ satisfies $\tau^2 = \rho$. The following proposition serves to establish a similar result for the Lorentz group. Recall that the map λ was defined as $\lambda(a,b) = j_a j_b$.

Proposition 4.4. *Suppose that* $\lambda(a,b)=\lambda(c,d)$*. Then:*

(i) $\lambda(a,c)=\lambda(b,d)$.

(ii) $\lambda(a,b)$ *and* $\lambda(a,c)$ *commute.*

(iii) There exists $\mu\in L_0$ *with* $[\mu,\lambda(a,b)]=1$ *and* $\mu^2(a,b)=\pm(c,d)$ *or* $\mu^2(a,b)=\pm(c,-d)$.

Proof of (i). $j_a j_c = j_a(j_c j_d)j_d = j_a(j_a j_b)j_d = j_b j_d$.

Proof of (ii). $j_a j_b\, j_c j_a\, j_b j_a = (j_a j_b)j_c(j_a j_b)j_a = j_c j_d j_c j_c j_d j_a = j_c j_a$.

Proof of (iii). In the previous step we proved $\lambda(a,b)$ and $\lambda(a,c)$ to commute. We now have to show that there exists a Lorentz transformation ν with $[\nu,\lambda(a,b)]=1$ and $\nu^2(a,b)=(\pm c,\pm d)$ for an arbitrary choice of signs. If $\lambda(a,b)=1$ or $\lambda(c,a)=1$, the statement is trivial. So assume $\lambda(a,b)\neq 1\neq\lambda(c,a)$.

By Lemma 4.2, there exist square roots ν_{ab} and ν_{cd} of $\lambda(a,b)$ and square roots ν_{ca} and ν_{db} of $\lambda(c,a)$ with

$$a=\pm\nu_{ab}b,\quad c=\pm\nu_{cd}d\quad\text{and}\quad c=\pm\nu_{ca}a,\quad d=\pm\nu_{db}b$$

for some choice of signs.

It suffices to prove $\nu_{cd}\nu_{db}=\nu_{db}\nu_{cd}$ and $\nu_{ab}b=\pm\nu_{cd}b$, since these relations yield the statement by

$$(c,d)=(\pm\nu_{cd}\nu_{db}b,\pm\nu_{db}b)=\nu_{db}(\pm\nu_{cd}b,\pm b)=\nu_{db}(\pm\nu_{ab}b,\pm b)=\nu_{db}(\pm a,\pm b).$$

If $\lambda(a,b)^2\neq 1$, one obtains $\nu_{cd}\nu_{db}=\nu_{db}\nu_{cd}$ from statement (ii) and Lemma 4.3 (iii). The remaining condition $\nu_{ab}b=\pm\nu_{cd}b$ then follows from

$$j_{\nu_{ab}b}=j_c(j_c j_a)=j_c(j_d j_b)=j_c(\nu_{cd}(j_d j_b)\nu_{cd}^{-1})=j_c(j_c j_{\nu_{cd}b}))=j_{\nu_{cd}b}.$$

If $\lambda(a,b)^2=1$, one obtains $b=\pm\lambda(a,b)b$ from $1=j_a j_b\, j_a j_b=j_{-j_a b}j_b$. Lemma 4.3.$(i)$ implies $\nu_{ab}^{-1}\nu_{cd}=\lambda(a,b)$ or $\nu_{ab}^{-1}\nu_{cd}=1$, proving $\nu_{ab}b=\pm\nu_{cd}b$. The proof is completed by observing

$$\nu_{cd}\lambda(d,b)\nu_{cd}^{-1}=j_c j_{\nu_{cd}b}=j_c j_{\nu_{ab}b}=j_c j_a=\lambda(d,b)$$

and an application of Lemma 4.3.(iii) yielding $\nu_{cd}\nu_{db}=\nu_{db}\nu_{cd}$. Since $\nu_{db}^2\neq 1$, any square root ν of ν_{db} commutes with $\nu_{cd}^2=\lambda(c,d)$ serving as the element with the desired properties. □

These observations are important for the characterisations of the universal cover of the Lorentz group by pairs of wedges to be introduced in the subsequent section.

4.2 The Universal Cover of the Lorentz Group

In the last section it was shown that every Lorentz transformation can be represented as the product of two reflections in the edges of wedges. The goal is now to find a description of the universal covering group of L_0 in terms of pairs of wedges. One of the main results of this thesis is the realisation called $\mathbf{G}_L$ which has been given in [KL07]. It is based on ideas which entered in Section 3.2.1 the construction of $\mathbf{G}_R$, the universal cover of $SO(3)$. Another important approach will be given later in Section 4.2.3, making use of the spin group $Spin(\mathbb{R}^{1+3})_0$ introduced in Section 2.4.1. The construction of both realisations of the universal cover of the Lorentz group is motivated by the problem of defining a representation of $\tilde{L}_+^\uparrow$ in terms of pairs of modular conjugations.

4.2.1 The Construction of the Group G_L

In Section 3.2.1 an equivalence relation on the set of pairs of unit vectors in $\mathbb{R}^3$ was introduced (cf. [Kuc05]). The equivalence relation was adjusted in such a way that the quotient space $\mathbf{G}_R$ is a model for the universal covering space of $SO(3)$. Since the Lorentz group is generated by pairs of reflections in edges of wedges, the goal is to introduce an equivalence relation on pairs of wedges so that the quotient space is a realisation of the universal covering of the Lorentz group.

The material we will introduce in this subsection concerning the construction of $\mathbf{G}_L$ has been published in [KL07, Sections 1.2 and 2]. Let us first fix the notation. Recall definition 2.3 of the set of *zweibeine* Z,

$$Z = \{\xi := (t_\xi, x_\xi) \in \mathbb{R}^{1+3} \times \mathbb{R}^{1+3};\ g(t_\xi, t_\xi) = 1, g(x_\xi, x_\xi) = -1, g(t_\xi, x_\xi) = 0\},$$

and that any zweibein $\xi = (t_\xi, x_\xi)$ defines a wedge W_ξ via

$$W_\xi := \{x \in \mathbb{R}^{1+3}; -g(x, x_\xi) > |g(x, t_\xi)|.\}$$

Two zweibeine ξ and χ are called equivalent, $\xi \bar{\sim} \chi$, if they generate the same wedge, i.e. if $W_\xi = W_\chi$. The canonical projection map associated with $\bar{\sim}$ is $\bar{\pi}$, and we define $\bar{Z} = Z/\bar{\sim}$. So the elements of $\bar{Z}$ are equivalence classes of zweibeine and $\bar{Z}$ describes the set of wedges whose edge contains the origin. It is therefore in one-to-one correspondence to $\mathcal{W}$.

For each $\xi \in Z$ let both j_ξ and $j_{\bar{\pi}(\xi)}$ denote the orthogonal reflection by the plane $\xi^\perp$, i.e., the map

$$j_\xi x \equiv j_{\bar{\pi}(\xi)} x := x - 2g(x, t_\xi)\, t_\xi + 2g(x, x_\xi)\, x_\xi.$$

Endow the set $\bar{Z} \times \bar{Z} =: \mathbf{M}_L$ with the structure of the pair groupoid of $\bar{Z}$ with concatenation $\circ$ and define an operation of L on $\mathbf{M}_L$ by $\mu(a, b) := (\mu a, \mu b)$. With each $(a, b) \in \mathbf{M}_L$ one can associate the Lorentz transformation

$$\lambda(a, b) := j_a j_b \in L_0. \tag{4.3}$$

$\mathbf{M}_L$ is now the analogue of $S^2 \times S^2$ in 3.2.1. The next step is to find an equivalence relation on $\mathbf{M}_L$ so that $\mathbf{M}_L/\sim$ is the universal cover of L_0. This turns out to be much more involved than for the rotation group. The equivalence relation defined there was based upon the notion of axis for a rotation for which no equivalent is at hand in the Lorentz group.

So define a relation $\sim$ on $\mathbf{M}_L$ by writing $(a,b) \sim (c,d)$ if and only if there exists $\mu \in L_0$ with $(c,d) = \pm\mu^2(a,b)$ and $\mu\lambda(a,b)\mu^{-1} = \lambda(a,b)$. Note that $(a,b) \sim (c,d)$ implies $\lambda(a,b) = \lambda(c,d)$.

Remark. Translated to the case of rotations, this definition coincides with definition 3.1 on page 44. Two rotations σ and ρ have the same axis if and only if there is a rotation τ commuting with σ and satisfying $\tau^2 = \sigma$.

The following results establish that the relation $\sim$ is an equivalence relation and that the quotient space indeed is a realisation of the universal cover of the restricted Lorentz group. The proofs, which are based on a couple of lemmas, will be presented in the following paragraph.

Proposition 4.5. *The relation $\sim$ is an equivalence relation.*

The proof will be given on page 60. The proposition allows to consider the quotient space of the equivalence relation.

Let $\mathbf{G}_L$ be the quotient space $\mathbf{M}_L/\sim$ and denote the canonical projection of the relation $\sim$ by π. Define $\pm 1 := \pi(a, \pm a)$ for arbitrary $a \in \bar{Z}$ and $-\pi(a,b) := \pi(a,-b)$ for $(a,b) \in \mathbf{M}_L$. As remarked before $\mu \sim \nu$ implies $\lambda(\mu) = \lambda(\nu)$, so a map $\tilde{\lambda} : \mathbf{G}_L \to L_0$ can be defined by $\tilde{\lambda}(g) := \lambda(\pi^{-1}(g))$, and the diagram

$$\begin{array}{ccc} \mathbf{M}_L & \xrightarrow{\pi} & \mathbf{G}_L \\ {\scriptstyle\lambda}\downarrow & \swarrow{\scriptstyle\tilde{\lambda}} & \\ L_+^\uparrow & & \end{array} \tag{4.4}$$

commutes. All maps in this diagram are continuous. This holds for π by definition, and it is evident for λ. To show continuity of $\tilde{\lambda}$, let $M \subset L_0$ be open. $\tilde{\lambda}^{-1}(M)$ is open if and only if $\pi^{-1}(\tilde{\lambda}^{-1}(M))$ is open. This set coincides with $\lambda^{-1}(M)$, which is open by continuity of λ.

The quotient space $\mathbf{G}_L$ is the candidate for a universal cover of the restricted Lorentz group and $\tilde{\lambda}$ is the candidate for a covering map. The next proposition states that the fibre of $\tilde{\lambda}$ over every Lorentz transformation contains precisely two elements, as expected for a two-sheeted covering map.

Proposition 4.6. *For each $g \in \mathbf{G}_L$, one has $g \neq -g$ and $\tilde{\lambda}^{-1}(\tilde{\lambda}(g)) = \{g, -g\}$.*

The proof of this proposition will be given in on page 61. The main theorem in this section states that $\mathbf{G}_L$ and $\tilde{\lambda}$ have the necessary properties of a universal covering group of the restricted Lorentz group and a covering map, respectively.

Theorem 4.1.

(i) $\tilde{\lambda}$ *is a covering map and endows* $\mathbf{G}_L$ *with the structure of a two-sheeted covering space of* L_0*.*

(ii) $\mathbf{G}_L$ *is simply connected.*

(iii) There is a unique group product $\odot$ *on* $\mathbf{G}_L$ *with the property that the diagram*

$$\begin{array}{ccc} \mathbf{M}_L \times \mathbf{M}_L & \xrightarrow{\circ} & \mathbf{M}_L \\ \downarrow{\scriptstyle \pi\times\pi} & & \downarrow{\scriptstyle \pi} \\ \mathbf{G}_L \times \mathbf{G}_L & \xrightarrow{\odot} & \mathbf{G}_L \\ \downarrow{\scriptstyle \tilde{\lambda}\times\tilde{\lambda}} & & \downarrow{\scriptstyle \tilde{\lambda}} \\ L_1 \times L_1 & \xrightarrow{\cdot} & L_1 \end{array} \tag{4.5}$$

commutes.

So $\mathbf{G}_L$ is isomorphic to the universal covering group of L_0. The proof of this theorem will be given in the paragraph starting on page 65. An important property of the group product in $\mathbf{G}_L$ is formulated in the subsequent lemma whose proof is given on page 72.

Lemma 4.7 (Adjoint action of $\mathbf{G}_L$ on itself). *Given* $h \in \mathbf{G}_L$ *and* $(c, d) \in \mathbf{M}_L$ *one has*

$$h\pi(c,d)h^{-1} = \pi\left(\tilde{\lambda}(h)c, \tilde{\lambda}(h)d\right). \tag{4.6}$$

4.2.2 The Construction of $\mathbf{G}_L$: Proofs

In the preceding section it has been stated that $\mathbf{G}_L$ is well-defined and a realisation of the universal covering space of $\tilde{L}_+^\uparrow$. The somewhat lengthy proofs will be given now. Proposition 4.5 states that the relation $\sim$ on pairs of wedges is an equivalence relation. In contrast to the corresponding statement for the analysis of the rotation group and its universal cover (see Section 3.2.1) this is not self-evident. It will be proven first in this section together with some geometrical properties of the equivalence relation $\sim$. These imply that the candidate $\tilde{\lambda}$ for the covering map is two-to-one, as required.

Next some topological features of the set $\bar{Z}$, namely first-countability and the Hausdorff property will be established. This is allows for using sequences and their unique limit points to proving sets to be open and is essential Hausdorff property of $\mathbf{G}_L$. This property and $\tilde{\lambda}$ being a covering map will first be checked on an open subset of $\mathbf{G}_L$ by defining an homeomorphism to some other Hausdorff space. By varying the reference frame via the choice of the timelike vector e_0, these properties can be extended to the whole of $\mathbf{G}_L$ up to the two elements $\tilde{\lambda}^{-1}(1)$. For this purpose, an observation concerning the polar decomposition of Lorentz transformations is required. Namely, one can always find a reference frame in which the rotation and boost in the decomposition do not commute. Finally, the Hausdorff property for the remaining elements is proven, as well as the fact that $\tilde{\lambda}$ is a covering map.

Proposition 4.5. $\sim$ *is an equivalence relation.*

Proof. Symmetry and reflexivity are evident, so it remains to prove transitivity. If $\underline{m} \sim \underline{n}$ and $\underline{n} \sim \underline{r}$, then $\lambda(\underline{m}) = \lambda(\underline{n}) = \lambda(\underline{r}) =: \lambda$, and there exist elements μ and ν commuting with λ and satisfying $\mu^2\underline{m} = \pm\underline{n}$ and $\nu^2\underline{n} = \pm\underline{r}$. If $\mu^2 = 1$ or $\nu^2 = 1$, one trivially has $\underline{m} \sim \underline{r}$. If $\nu^2\mu^2 = 1$, one even has $\underline{m} = \pm\underline{r}$. It follows from $\nu^2\mu^2\underline{m} = \pm\underline{r}$ that

$$\lambda = j_{\nu^2\mu^2 a} j_{\nu^2\mu^2 b} = \nu^2\mu^2 j_a j_b \mu^{-2}\nu^{-2} = \nu^2\mu^2\lambda\mu^{-2}\nu^{-2},$$

and one concludes from Lemma 4.3.(ii) that there exists a square root κ of $\nu^2\mu^2$ commuting with λ. □

Recall that the map λ, defined in eq. (4.3), maps a pair of wedges to the Lorentz transformation given by the product of the reflections in the edges of the wedges. The following lemmas serve as a preparation for the proof of Proposition 4.6. First it is verified that the fibre over a Lorentz transformation contains at least two elements. Then it is shown that any pair of wedges generating the same Lorentz transformation as a pair (a, b) belongs either to the equivalence class of (a, b) or to the class of $(a, -b)$.

Lemma 4.8. $(a, b) \not\sim (a, -b)$ *for* $(a, b) \in \mathbf{M}_L$*, i.e.,* $g \neq -g$ *for all* $g \in \mathbf{G}_L$*.*

Proof. The statement is evident for $b = \pm a$, so it remains to consider the case $\lambda(a, b) \neq 1$. Assume $(a, b) \sim (a, -b)$. By Lemma 4.2, there exists an element $\mu \in L_0$ with $\mu^2 = \lambda(a, b)$ and $(a, b) = (\pm\mu b, b)$, and, by assumption, there exists an element $\nu \in L_0$ with $\nu\mu^2\nu^{-1} = \mu^2$ and $\nu^2(a, b) = \pm(a, -b)$. μ^2 and ν^2 commute and differ from 1, so μ and ν commute by Lemma 4.3.(iii). Assume without loss of generality that $a = \mu b$, then one obtains

$$(a, b) = (\mu b, b) = \pm\nu^{-2}(\mu b, -b) = \pm(\mu\nu^{-2}b, -\nu^{-2}b) = \pm(-\mu b, b) = \pm(-a, b),$$

leading to the contradiction $a = -a$ or $b = -b$, respectively. □

Lemma 4.9. *Suppose that* $\lambda(a, b) = \lambda(c, d)$*. Then* $(a, b) \sim (c, d)$ *or* $(a, b) \sim (c, -d)$*.*

Proof. Recall from Lemma 4.4 that $\lambda(a, c) = \lambda(b, d)$ and that $\lambda(a, b)$ and $\lambda(a, c)$ commute. Since, by definition, $(a, b) \sim (-a, -b)$, it suffices to prove $(a, b) \sim (\pm c, \pm d)$ for an arbitrary choice of signs. If $\lambda(a, b) = 1$ or $\lambda(c, a) = 1$ the statement is trivial. So assume $\lambda(a, b) \neq 1 \neq \lambda(c, a)$.

By Lemma 4.2, there exist square roots ν_{ab} and ν_{cd} of $\lambda(a,b)$ and square roots ν_{ca} and ν_{db} of $\lambda(c,a)$ with

$$a = \pm\nu_{ab}b, \quad c = \pm\nu_{cd}d \quad \text{and} \quad c = \pm\nu_{ca}a, \quad d = \pm\nu_{db}b$$

for some choice of signs.

It suffices to prove $\nu_{cd}\nu_{db} = \nu_{db}\nu_{cd}$ and $\nu_{ab}b = \pm\nu_{cd}b$, since these relations yield the statement by

$$(c,d) = (\pm\nu_{cd}\nu_{db}b, \pm\nu_{db}b) = \nu_{db}(\pm\nu_{cd}b, \pm b) = \nu_{db}(\pm\nu_{ab}b, \pm b) = \nu_{db}(\pm a, \pm b).$$

If $\lambda(a,b)^2 \neq 1$, one obtains $\nu_{cd}\nu_{db} = \nu_{db}\nu_{cd}$ from statement (ii) and Lemma 4.3 (iii). The remaining condition $\nu_{ab}b = \pm\nu_{cd}b$ then follows from

$$j_{\nu_{ab}b} = j_c(j_c j_a) = j_c(j_d j_b) = j_c(\nu_{cd}(j_d j_b)\nu_{cd}^{-1}) = j_c(j_c j_{\nu_{cd}b})) = j_{\nu_{cd}b}.$$

If $\lambda(a,b)^2 = 1$, one obtains $b = \pm\lambda(a,b)b$ from $1 = j_a j_b\, j_a j_b = j_{-j_a b} j_b$. Lemma 4.3.$(i)$ implies $\nu_{ab}^{-1}\nu_{cd} = \lambda(a,b)$ or $\nu_{ab}^{-1}\nu_{cd} = 1$, proving $\nu_{ab}b = \pm\nu_{cd}b$. The proof is completed by observing

$$\nu_{cd}\lambda(d,b)\nu_{cd}^{-1} = j_c j_{\nu_{cd}b} = j_c j_{\nu_{ab}b} = j_c j_a = \lambda(d,b)$$

and an application of Lemma 4.3.(iii) yielding $\nu_{cd}\nu_{db} = \nu_{db}\nu_{cd}$. □

One now immediately obtains from Lemmas 4.8 and 4.9

Proposition 4.6. *For each* $g \in \mathbf{G}_L$ *the fibre* $\tilde{\lambda}^{-1}(\tilde{\lambda}(g))$ *contains precisely two elements.*

Proof. $g \neq -g$ and $\tilde{\lambda}(g) = \tilde{\lambda}(-g)$ for all g, so each $\tilde{\lambda}^{-1}(\tilde{\lambda}(g))$ contains at least two elements.

By construction, one has $\lambda(a,b) = \tilde{\lambda}(g)$ for each $(a,b) \in \pi^{-1}(g)$. If $(c,d) \in \mathbf{M}_L$ satisfies $\lambda(c,d) = \tilde{\lambda}(g) = \lambda(a,b)$ as well, Lemma 4.9 implies that $(a,b) \sim (c,d)$ or $(a,b) \sim (c,-d)$, so $\tilde{\lambda}^{-1}(\tilde{\lambda}(g))$ contains at most two elements. □

The first property of $\tilde{\lambda}$, the candidate for the covering map, has been established: The preimage of every Lorentz transformation contains precisely two elements, as expected for a two-sheeted covering map.

The sets Z and $\bar{Z}$

The next step in the proof of Theorem 4.1, stating that $\tilde{\lambda}$ is indeed a covering map and that $\mathbf{G}_L$ is simply connected, is taken. The topological properties of the sets Z of zweibeine and $\bar{Z}$, the set of equivalence classes describing wedges, will be worked out in the following.

The zweibeine in

$$Z^+ := \{\xi \in Z : t_\xi \in V^+\} \subset Z \tag{4.7}$$

suffice to describe all wedges, i.e. $\{W_\xi;\, \xi \in Z^+\} = \{W_\xi;\, \xi \in Z\}$. The latter set, the set of wedges, was denoted by $\mathcal{W}$.

Lemma 4.10. *The actions of L_0 on Z^+ and of L on Z are transitive.*

Proof. The Lorentz group L acts transitively on the set of orthonormal frames in Minkowski space, so the latter statement is trivial since the zweibeine can always be extended to orthonormal frames. For two elements in Z^+, the Lorentz transformation can be chosen in L_+, because we can complete the zweibeine in Z^+ to orthonormal frames. If one has a Lorentz transformation in L_- connecting these, then changing the orientation of one of the orthonormal frames the new connecting Lorentz transformation can be chosen in L_+. It is also in $L_0 = L_+^\uparrow$ because both timelike vectors in the orthonormal frames are future directed. □

It is desirable to establish first-countability and the Hausdorff property for the topology of $\bar{Z}$. These are not inherited to the quotient space from the base space. Without first-countability one is enforced to consider nets instead of sequences to have a sensible notion of limit at hand.

Lemma 4.11. *$\bar{Z}$ is a first-countable topological space.*

Proof. Let H be a Cauchy surface. Then the set $Z_H := \{\xi \in Z^+; x_\xi \in H\}$ is a closed subset of Z^+.

For each $\xi \in Z^+$ the intersection of the inextendible curve Γ_ξ, whose values are given by $\{x_\chi; W_\chi = W_\xi\}$, with H contains precisely one element y_ξ, and there is a unique generalised boost $\beta_H(\xi)$ with $y_\xi = \beta_H(\xi)x_\xi$.

Define a map $\zeta_H : Z^+ \to Z_H$ by $\zeta_H(\xi) := \beta_H(\xi)\xi$. Then $\xi \tilde{\sim} \chi$ implies $\zeta_H(\xi) = \zeta_H(\chi)$ by construction, so a map $\bar{\zeta} : \bar{Z} \to Z_H$ is well-defined by $\bar{\zeta}(\bar{\pi}(\xi)) = \zeta(\xi)$. The diagram

$$\begin{array}{ccc} Z^+ & \xrightarrow{\bar{\pi}} & \bar{Z} \\ {\scriptstyle \zeta_H}\downarrow & \swarrow {\scriptstyle \bar{\zeta}_H} & \\ Z_H & & \end{array}$$

commutes. All maps in this diagram are continuous. This holds for $\bar{\pi}$ by definition, and it is evident for ζ_H. To show continuity of $\bar{\zeta}_H$, let $M \subset L_0$ be open. $\bar{\zeta}_H^{-1}(M)$ is open if and only if $\pi^{-1}(\bar{\zeta}_H^{-1}(M))$ is open. This set coincides with $\zeta_H^{-1}(M)$, which is open by continuity of ζ_H.

Since $\bar{\zeta}_H$ has the continuous inverse $\bar{\pi}|_{Z_H}$, one finds that Z_H and $\bar{Z}$ are homeomorphic topological spaces. Since Z_H is first-countable, so is $\bar{Z}$. □

One immediately concludes the following corollary.

Corollary 4.12. *$\bar{Z}$ and $\mathbf{M}_L$ are Hausdorff spaces.*

Polar decompositions on L_0

Further preparations are required for the proof of Theorem 4.1, which is much more involved than its prototype in Section 3.2.1 and Ref. [Kuc05]. A crucial tool will be the decomposition of Lorentz transformations into rotations and boosts. We will prove that for every nontrivial Lorentz transformation one can find a reference frame so that the rotation and boost in the decomposition do not commute.

Specify a time direction by choosing a future-directed timelike unit vector e_0. Consider the Euclidean inner product $\langle\cdot,\cdot\rangle_{e_0}$ on $\mathbb{R}^{1+3}$ defined by $\langle x,y\rangle_{e_0} := -g(x,y)+2g(x,e_0)g(y,e_0)$. Denote the adjoint of a linear map $T:\mathbb{R}^{1+3}\to\mathbb{R}^{1+3}$ with respect to this inner product by $T^\dagger$. If T is a restricted Lorentz transformation, then the positive operator $\hat\beta(T) := |T| := (T^\dagger T)^{1/2}$ is a boost, and the orthogonal operator $\hat\rho := T\cdot|T|^{-1} = T\hat\beta(T)^{-1}$ is a rotation; $\hat\beta(T)$ and $\hat\rho(T)$ yield the polar decomposition $T=\hat\rho(T)\hat\beta(T)$ of T. This well-known fact has been discussed in [Mor02, Urb02]. The polar decomposition depends on the choice of the vector e_0 defining the time direction. On $\mathbf{G}_L$ define $\tilde\rho(g) := \hat\rho(\tilde\lambda(g))$ and $\tilde\beta(g) := \hat\beta(\tilde\lambda(g))$. For each $x\in\mathbb{R}^{1+3}$ denote the stabiliser of x in L_0 by $\mathfrak{S}(x) := \{\mu\in L_0 : \mu x = x\}$, and for each subset M of $\mathbb{R}^{1+3}$ define $\mathfrak{S}(M) := \bigcap_{x\in M}\mathfrak{S}(x)$.

To each time-zero unit vector e, assign the class $\bar e := \bar\pi(e_0,e)$. The following lemma establishes a second property of $\tilde\lambda$. It is an immediate consequence of Lemma 2.1 in Ref. [BS04]; whose proof is recalled here for the reader's convenience.

Lemma 4.13. *$\tilde\lambda$ is surjective.*

Proof. We prove that λ is surjective, then the statement follows. $\lambda(a,\pm a)=1$ for all $a\in\bar Z$, so it remains to show that $\lambda^{-1}(\mu)\neq\emptyset$ for each $\mu\neq 1$.

Suppose that $\mu =: \rho$ is a rotation, that τ is a root of ρ and that e is a time-zero unit vector in the rotation plane of ρ. Then $\rho = \rho j_{\bar e} j_{\bar e} = j_{\tau\bar e} j_{\bar e} = \lambda(\tau\bar e,\bar e)$.

Suppose that $\mu =: \beta$ is a boost, and let e be a time-zero unit vector in the fixed-point set of β. Then $\beta = j_{\bar e} j_{\bar e}\beta = j_{\bar e} j_{\beta^{-1/2}\bar e} = \lambda(\bar e,\beta^{-1/2}\bar e)$.

In the remaining case that both $\hat\rho(\mu)$ and $\hat\beta(\mu)$ differ from 1 the rotation plane of $\hat\rho(\mu)$ and the fixed-point plane of $\hat\beta(\mu)$ are well-defined two-dimensional planes contained in the time-zero plane. Since the time-zero plane is three-dimensional, this implies that the intersection of these planes is nonempty. Let e be a unit vector in this intersection and let τ be a root of $\hat\rho(\mu)$. Then $\mu = \hat\rho(\mu) j_{\bar e} j_{\bar e}\hat\beta(\mu) = j_{\tau\bar e} j_{\hat\beta(\mu)^{-1/2}\bar e} = \lambda(\tau\bar e,\beta^{-1/2}(\mu)\bar e)$. □

Denote the set of rotations by R and the set of boosts by B. Furthermore, define $\dot R := R\setminus\{1\}$ and $\dot B := B\setminus\{1\}$ and write $\ddot R := \{\sigma\in R : \sigma^2\neq 1\}$. The dependence of the polar decomposition on the time direction given by e_0 will be of importance. In particular, it is useful to find a frame in which the rotation and boost part of the polar decomposition do not commute. The following lemmas are concerned with this question, which will be finally answered with lemma 4.16 stating that this is always possible for any nontrivial Lorentz transformation.

Lemma 4.14. *$\rho \in \dot{R}$ and $\beta \in \dot{B}$ commute if and only if* $\mathrm{FP}(\rho) = \mathrm{FP}(\beta)^\perp$.

Proof. Assume $\rho\beta = \beta\rho$. If $x \in \mathrm{FP}(\beta)$, then $\beta\rho x = \rho\beta x = \rho x$, so $\rho\,\mathrm{FP}(\beta) = \mathrm{FP}(\beta)$, whence one concludes that either $\mathrm{FP}(\beta) = \mathrm{FP}(\rho)$ or $\mathrm{FP}(\beta) = \mathrm{FP}(\rho)^\perp$. Since $\mathrm{FP}(\beta)$ is a spacelike surface, whereas $\mathrm{FP}(\rho)$ is timelike, one concludes $\mathrm{FP}(\beta) = \mathrm{FP}(\rho)^\perp$. That the condition is sufficient is trivial. □

Lemma 4.15.

(i) Consider $\mu \in L_0$ with polar decomposition $\mu = \rho\beta$. Then $\rho\beta = \beta\rho$ if and only if there exists a time-zero unit vector e with $\mu \in \mathfrak{S}(\bar{e})$.

(ii) Given $a, b \in \bar{Z}$, one has $\mathfrak{S}(a) \cap \mathfrak{S}(b) \neq \{1\}$ if and only if $a = \pm b$.

Proof of (i). Each rotation or boost is contained in the stabiliser of $\bar{e}$ for some e, so statement (i) trivially holds for rotations or boosts.

It remains to consider the case that $\rho \neq 1 \neq \beta$. If $\rho\beta = \beta\rho$, then it follows from Lemma 4.14 that the rotation axis of ρ is parallel to the boost direction of β. Let e be one of the two unit vectors on this axis, then ρ, β and also $\rho\beta$ are contained in $\mathfrak{S}(W_{\bar{e}}) = \mathfrak{S}(\bar{e})$. So the condition is necessary.

If, conversely, $\mu \in \mathfrak{S}(\bar{e})$, then there exists a unique boost γ with $\gamma(\mu e_0, \mu e) = (e_0, e)$ and $\gamma \in \mathfrak{S}(\bar{e})$, because $\gamma\bar{e} = \gamma\mu\bar{e} = \bar{e}$. Since $\mathfrak{S}(\bar{e})$ is abelian, $\gamma\mu = \mu\gamma = \rho\beta\gamma$. The product $\rho\beta\gamma$ has the fixed points e_0 and e by definition of γ, so it is a rotation, and $\beta\gamma = 1$ by uniqueness of the polar decomposition. As seen above γ commutes with μ, so β^{-1} commutes with $\rho\beta$, i.e., $\rho = \beta^{-1}\rho\beta$.

Proof of (ii). Without loss of generality, assume $a = \bar{e}$. If μ is a rotation, then e is on the axis of rotation of μ, so $\mathrm{FP}(\mu) = \bar{e}^{\perp\perp}$, and the plane $\bar{e}^\perp$ is mapped onto itself. The only other time-zero unit vector on the axis of μ is $-e$, so $b = \pm\bar{e} = \pm a$ as stated.

If μ is a boost, then the vectors $\ell^+ := e + e_0$ and $\ell^- := e - e_0$ are eigenvectors of μ associated with distinct eigenvalues ε and ε^{-1}. The vectors $\ell^\pm$ are perpendicular to $\mathrm{FP}(\mu)$ by invariance of the metric: If $x \in \mathrm{FP}(\mu)$, then

$$\varepsilon\, g(x, \ell^+) = g(x, \mu\ell^+) = g(\mu^{-1}x, \ell^+) = g(x, \ell^+),$$

so $\varepsilon \neq 1$ implies $g(x, \ell^+) = 0$, and one obtains $\mathrm{FP}(\mu) = \bar{e}^\perp$.

It remains to consider the case that $\rho \neq 1 \neq \beta$. By Lemma 4.14, statement (i) implies $\mathrm{FP}(\rho) \perp \mathrm{FP}(\beta)$, so $\ell^\pm$ are fixed points of ρ and hence eigenvectors not only of β, but also of μ. Additional eigenvectors in $\bar{e}^\perp$ exist only if ρ is a rotation by the angle π; their eigenvalue is -1. Since $\varepsilon \neq -1 \neq \varepsilon^{-1}$, the vectors $\ell^\pm$ are the only eigenvectors of μ with positive eigenvalues.

By assumption, $\mu \in \mathfrak{S}(b) =: \mathfrak{S}(\bar{\pi}(f_0, f))$, so the polar decomposition of μ with respect to f_0 commutes. By the reasoning just used, the lightlike vectors $f + f_0$ and $f - f_0$ are eigenvectors of μ with positive eigenvalues and hence proportional to $e + e_0$ and $e - e_0$, respectively, whence $\bar{e} = \pm\bar{f}$ and statement (ii) is obtained. □

Recall that the set future-directed elements of the positive unit hyperboloid in Minkowski space is denoted by M_1^+.

Lemma 4.16. *Given any $\mu \in L_0$, suppose that the polar decomposition $\mu = \rho_{e_0}\beta_{e_0}$ commutes for all $e_0 \in M_1^+$. Then $\mu = 1$.*

Proof. Because, by assumption, $\rho_{e_0}\beta_{e_0} = \beta_{e_0}\rho_{e_0}$, there is some time-zero unit vector e with $\mu \in \mathfrak{S}(\bar{e})$. The subset

$$t_{\bar{e}} := \{d_0 \in M_1^+;\, \bar{\pi}(d_0, d) = \bar{e} \quad \text{for some unit vector } d \perp d_0\}$$

of M_1^+ is a hyperbola, so there exists some $f_0 \in M_1^+ \setminus t_{\bar{e}}$.

By assumption, the polar decomposition $\mu = \rho_{f_0}\beta_{f_0}$ commutes as well, so there is some unit vector $f \perp f_0$ with $\mu \in \mathfrak{S}(\bar{\pi}(f_0, f))$. By construction, $\bar{\pi}(f_0, f) \neq \pm\bar{e}$, so $W_{\bar{\pi}(f_0,f)} \neq \pm W_{\bar{e}}$, whence $\mathfrak{S}(\bar{\pi}(f_0, f)) \cap \mathfrak{S}(\bar{e}) = \{1\}$ by Lemma 4.15. □

For each $(\rho, \beta) \in \dot{R} \times B$, let $E(\rho, \beta)$ be the set of all time-zero unit vectors in $\mathrm{FP}(\rho)^\perp \cap \mathrm{FP}(\beta)$. In other words, $E(\rho, \beta)$ is the set of spacelike unit vectors orthogonal to e_0, the axis of rotation of ρ and the boost direction of β (if β is nontrivial). These vectors are the ones employed in the decomposition of a Lorentz transformation into two reflections in edges of wedges(cf. equation 4.2).

Proposition 4.17.

(i) $E(\rho, \beta) \cong S^1$ if and only if $\rho\beta = \beta\rho$.

(ii) Otherwise, $E(\rho, \beta) = \{\pm e\}$ for some time-zero unit vector e.

Proof of (i). If $\beta = 1$, then $E(\rho, \beta) = \mathrm{FP}(\rho)^\perp \cap (\{0\} \times S^2)$, i.e., the intersection of the time-zero two-sphere with a two-dimensional spacelike subspace of the time-zero plane. Such an intersection is homeomorphic to S^1. If $\rho \neq 1 \neq \beta$, then $\rho\beta = \beta\rho$ if and only if $\mathrm{FP}(\beta) \perp \mathrm{FP}(\rho)$ by Lemma 4.14, and this holds if and only if $\mathrm{FP}(\rho)^\perp \cap \mathrm{FP}(\beta)$ is a two-dimensional spacelike plane, i.e., if and only if $E(\rho, \beta)$ is homeomorphic to S^1.
Proof of (ii). If $\rho\beta \neq \beta\rho$, then $\mathrm{FP}(\rho)^\perp \cap \mathrm{FP}(\beta)$ is not two-dimensional by Lemma 4.14, but, since $\mathrm{FP}(\rho)^\perp$ and $\mathrm{FP}(\beta)$ are two-dimensional subspaces of the time-zero plane, their intersection is one-dimensional and contains two opposite time-zero unit vectors. □

Proof of Theorem 4.1

The central Theorem 4.1 stated that $\mathbf{G}_L$ is the universal (two-sheeted) covering group of the restricted Lorentz group. Its proof is split into several steps. The Hausdorff property and the necessary features for $\tilde{\lambda}$ to be a covering map are first established for an open subset $\mathbf{G}_L^{e_0}$ of $\mathbf{G}_L$. The results are extended to $\mathbf{G}_L \setminus \tilde{\lambda}^{-1}(1)$ by varying the reference frame specified by e_0.

Let N^{e_0} be the set of all $(\tau, \beta) \in \ddot{R} \times B$ with $E(\tau, \beta) \cong \mathbb{Z}_2$ (cf. prop. 4.17). So the elements of N^{e_0} are pairs of a rotation with square different from the identity and boosts with the property that the rotation axis is not parallel to the direction of the boost.

Define a map $\lambda_1 : N^{e_0} \to L_0$ by $\lambda_1(\tau, \beta) := \tau^2 \beta$ and define ${L_0}^{e_0} := \lambda_1(N^{e_0})$. Furthermore, set $\mathbf{G}_L^{e_0} := \tilde{\lambda}^{-1}(L_0^{e_0})$.

For each $\rho \in \ddot{R}$ there is a unique time-zero unit vector $\mathbf{a}(\rho)$ with the property that ρ is a right-handed rotation with respect to $\mathbf{a}(\rho)$ by a rotation angle $\alpha(\rho)$ smaller than π. The functions $\mathbf{a}(\cdot)$ and $\alpha(\cdot)$ are continuous on $\ddot{R}$, and α has a continuous extension to a function from all of R onto the closed interval $[0, \pi]$. We denote this extension by α as well.

For each $\beta \in \dot{B}$ there exists a unique time-zero unit vector $\mathbf{b}(\beta)$ with respect to which β is a boost by a rapidity $\chi(\beta)$ greater than zero. The functions $\mathbf{b}$ and χ are continuous, and the function χ has a continuous extension to all of B with values in $\mathbb{R}^{\geq 0}$, denoted by χ as well.

The functions $\tilde{\alpha} : \mathbf{G}_L \to [0, \pi]$ and $\tilde{\chi} : \mathbf{G}_L \to \mathbb{R}^{\geq 0}$ defined by $\tilde{\alpha}(g) := \alpha(\tilde{\rho}(g))$ and $\tilde{\chi}(g) := \chi(\tilde{\beta}(g))$ are continuous.

Lemma 4.18.

(i) The polar decomposition $\hat{\rho} \times \hat{\beta} : L_0 \to R \times B$ is continuous.

(ii) The restriction of the group product in L_0 to $R \times B$ is a homeomorphism onto L_0.

(iii) N^{e_0} is a two-sheeted covering space of $L_0^{e_0}$ when endowed with the covering map λ_1.

Proof of (i). The group product in L_0, the map $\mu \mapsto \mu^\dagger$ and the square-root function are continuous. The map $\mu \mapsto \hat{\beta}(\mu) := \sqrt{\mu^\dagger \mu}$ is continuous. Since the map $\mu \mapsto \mu^{-1}$ is continuous as well, one concludes that $\mu \mapsto \hat{\rho}(\mu) := \mu \hat{\beta}(\mu)^{-1}$ is continuous.

Proof of (ii). The group product is continuous and inverse to the continuous polar decomposition. Since the group product is onto, so is the polar decomposition.

Proof of (iii). N^{e_0} is an open subset of $\ddot{R} \times B$, so it suffices to prove the corresponding statement for $\ddot{R} \times B$. It remains to show that $\ddot{R}$ is a two-sheeted covering space when endowed with the covering map $\tau \mapsto \tau^2$. Continuity of this map follows from continuity of the group product. Conversely, each $\rho \in \ddot{R}$ has the two roots $[\mathbf{a}(\rho), \alpha(\rho/2)]$ and $[-\mathbf{a}(\rho), \pi - \alpha(\rho)/2]$, and, since $\mathbf{a}$ and α are continuous maps, the square map has a continuous local inverse. $\square$

The rotation-boost decomposition allows for characterising an element of $\mathbf{G}_L^{e_0}$ by a unique rotation, a unique boost and a spacelike unit vector which is unique up to a sign.

Lemma 4.19. *For each $g \in \mathbf{G}_L^{e_0}$ there is a unique square root $\tilde{\tau}(g)$ of $\tilde{\rho}(g)$ with $g = \pi(\tilde{\tau}(g)\bar{e}, \tilde{\beta}(g)^{-1/2}\bar{e})$ independent of the choice of e in the set $E(\tilde{\tau}(g), \tilde{\beta}(g))$ (which contains two elements).*

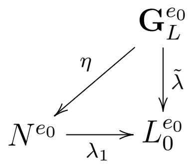
Proof. If $e \in \mathrm{FP}(\tilde{\beta}(g))$, then $\lambda(\bar{e}, \tilde{\beta}(g)^{-1/2}\bar{e}) = \tilde{\beta}(g)$. If $e \in \mathrm{FP}(\tilde{\rho}(g))^{\perp}$, there are precisely two $a \in \bar{Z}$ with $\lambda(a, \bar{e}) = \tilde{\rho}(g)$. Namely, if $\tau_{\pm}$ are the two square roots of the rotation $\tilde{\rho}(g)$, then $a_{\pm} = (\tau_{\pm}\bar{e}, \bar{e})$ have the desired property.

Accordingly, if $e \in E(\tilde{\rho}(g), \tilde{\beta}(g)) = \mathrm{FP}(\tilde{\rho}(g)) \cap \mathrm{FP}(\tilde{\beta}(g))^{\perp}$, the nonequivalent pairs $\underline{m}^{+}$ and $\underline{m}^{-}$ defined by

$$\underline{m}^{\pm} := (\tau_{\pm}\bar{e}, \bar{e}) \circ (\bar{e}, \tilde{\beta}(g)^{-1/2}\bar{e}) = (\tau_{\pm}\bar{e}, \tilde{\beta}(g)^{-1/2}\bar{e})$$

satisfy $\lambda(\underline{m}^{\pm}) = \tilde{\lambda}(g)$. By Proposition 4.6, exactly one of them is contained in $\pi^{-1}(g)$. □

Define a "polar decomposition" $\eta : \mathbf{G}_L^{e_0} \to N^{e_0}$ by $\eta(g) := (\tilde{\tau}(g), \tilde{\beta}(g))$. Evidently, η is a bijection, and the diagram

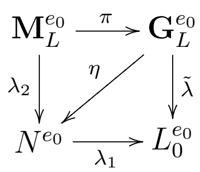


(4.8)

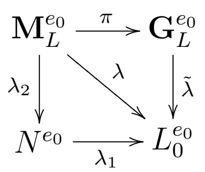
commutes. It will be shown that η is continuous and open, hence a homeomorphism. Since its range is a Hausdorff space, this also holds for $\mathbf{G}_L^{e_0}$. Define the subset

$$\mathbf{M}_L^{e_0} := \{(\tau\bar{e}, \beta^{-1/2}\bar{e});\ (\tau, \beta) \in N^{e_0},\ e \in E(\tau, \beta)\}$$

of $\mathbf{M}_L$ and define a map $\lambda_2 : \mathbf{M}_L^{e_0} \to N^{e_0}$ by $\lambda_2(\underline{m}) := \eta(\pi(\underline{m}))$. Then the diagrams



(A) and



(B) (4.9)

commute. Define a continuous function $\mathbf{e}$ from N^{e_0} to the spacelike unit vectors orthogonal to e_0 by

$$\mathbf{e}(\rho, \beta) := \frac{\mathbf{a}(\rho) \times \mathbf{b}(\beta)}{|\mathbf{a}(\rho) \times \mathbf{b}(\beta)|},$$

where $\times$ denotes the vector product within the time-zero plane $e_0^{\perp}$.

Lemma 4.20.

(i) $\lambda_{e_0} := \lambda|_{\mathbf{M}_L^{e_0}}$ *is an open map.*

(ii) λ_2 *is continuous.*

(iii) η *is continuous.*

Proof of (i). $L_0^{e_0}$ is first-countable, so it suffices to show that for each sequence $(\mu_n)_n$ in $L_0^{e_0}$ converging to a limit μ and for each $\underline{m} \in \lambda_{e_0}^{-1}(\mu)$ there exists a sequence $(\underline{m}_n)_n$ converging to $\underline{m}$ and satisfying $\lambda_{e_0}(\underline{m}_n) = \mu_n$.

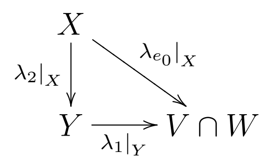
So let $(\mu_n)_n$ be a sequence in $L_0^{e_0}$ converging to μ. Then $(\hat{\rho}(\mu_n), \hat{\beta}(\mu_n))$ converges to $(\hat{\rho}(\mu), \hat{\beta}(\mu))$ in N^{e_0} by continuity of the functions $\hat{\rho}$ and $\hat{\beta}$. Consequently, the time-zero unit vectors $e_n := \mathbf{e}(\hat{\rho}(\mu_n), \hat{\beta}(\mu_n))$ tend to the limit $e = \mathbf{e}(\hat{\rho}(\mu), \hat{\beta}(\mu))$. Since $\bar{\pi}$ is continuous, the sequence $\bar{e}_n$ converges to $\bar{e}$.

Consider, without loss of generality, the element $\underline{m} := (\tau\bar{e}, \hat{\beta}(\mu)^{-1/2}\bar{e})$ of the fibre $\lambda^{-1}(\mu)$. There exists a convergent sequence $(\tau_n)_n$ in R with $\tau_n^2 = \hat{\rho}(\mu_n)$, and the sequence $(\underline{m}_n)_n$ defined by $\underline{m}_n := (\tau_n\bar{e}_n, \hat{\beta}(\mu_n)^{-1/2}\bar{e}_n)$ satisfies $\lambda_{e_0}(\underline{m}_n) = \mu_n$ and $\underline{m}_n \to \underline{m}$. The same reasoning applies to the other elements of the fibre $\lambda_{e_0}^{-1}(\mu)$.

Proof of (ii). For each $\underline{m}_1 \in \mathbf{M}_L^{e_0}$, the fibre $\lambda_{e_0}^{-1}(\lambda_{e_0}(\underline{m}_1))$ contains four elements $\underline{m}_1, \ldots, \underline{m}_4$, and, by the Hausdorff property, these have mutually disjoint open neighbourhoods $U_1, \ldots, U_4$. Since λ_{e_0} is open by statement (i), their images are open, so $V := \lambda_{e_0}(U_1) \cap \cdots \cap \lambda_{e_0}(U_4)$ is open.

On the other hand, there is an open neighbourhood Y of $\lambda_2(\underline{m}_1)$ with the property that $\lambda_1|_Y$ is a homeomorphism onto $W := \lambda_1(Y)$. Being a covering map, λ_1 is open, so W is open.

$V \cap W$ is open and λ_{e_0} is continuous, so the set $X := U_1 \cap \lambda_{e_0}^{-1}(V \cap W)$ is open and contains $\underline{m}_1$. The diagram



is a commutative diagram of bijections by construction. Since $\lambda_{e_0}|_X$ and $\lambda_1|_Y$ are homeomorphisms, so is $\lambda_2|_X$.

Proof of (iii). Using diagram 4.9 (B), one concludes the statement from continuity of λ_2. □

Lemma 4.21. $\mathbf{M}_L^{e_0}$ *is a two-sheeted covering space of* N^{e_0} *when endowed with the covering map* λ_2.

Proof. Define continuous maps $\underline{m}_\pm : N^{e_0} \to \mathbf{M}_L^{e_0}$ by

$$\underline{m}_\pm(\tau,\beta) := (\pm\tau\overline{\mathbf{e}(\tau,\beta)}, \pm\beta\overline{\mathbf{e}(\tau,\beta)}). \tag{4.10}$$

We show that these functions are local inverses of λ_2.

For a given $x \in N^{e_0}$ write $y_\pm := \underline{m}_\pm(x)$. Since $\mathbf{M}_L^{e_0}$ is a Hausdorff space, there exist two disjoint open neighbourhoods $Y_\pm$ of $y_\pm$. By continuity of $\underline{m}_\pm$, the pre-images $X_\pm := \underline{m}_\pm^{-1}(Y_\pm)$ are open, and $X := X_+ \cap X_-$ is an open neighbourhood of x. By continuity of λ_2, the sets $W_\pm := \lambda_2^{-1}(X) \cap Y_\pm$ are open neighbourhoods of $\underline{m}_\pm(x)$ with $\lambda_2(W_+) = X = \lambda_2(W_-)$. As a consequence, the continuous maps $\underline{m}_\pm|_X : X \to W_\pm$ are one-to-one and onto, their inverses being λ_2. □

Proposition 4.22.

(i) *η is a homeomorphism.*

(ii) *$\mathbf{G}_L^{e_0}$ is a Hausdorff space.*

(iii) *$\mathbf{G}_L^{e_0}$ is a two-sheeted covering space of $L_0^{e_0}$ when endowed with the covering map $\tilde{\lambda}_{e_0}$.*

Proof. Consider the map $\underline{m}_\pm$ defined in equation (4.10). The maps $\pi \circ \underline{m}_+$ and $\pi \circ \underline{m}_-$ coincide and are inverse to η by construction. By continuity of $\underline{m}_\pm$ and π, they are continuous. This proves (i) and implies (ii).

$\tilde{\lambda}_{e_0} = \lambda_2 \circ \eta$ is a concatenation of a homeomorphism and a two-sheeted covering map. This yields (iii). □

So the Hausdorff property holds for $\mathbf{G}_L^{e_0}$ and $\tilde{\lambda}_{e_0}$ is a covering map for this space. Next these results will be extended to $\dot{\mathbf{G}}_L$ and finally to $\mathbf{G}_L$. To this end recall that $\mu \in L_0^{e_0}$ if and only if $\hat{\rho}(\mu)\hat{\beta}(\mu) \neq \hat{\beta}(\mu)\hat{\rho}(\mu)$. Lemma 4.16 guarantees that for a nontrivial Lorentz transformation there always exists a reference frame specified by some e_0 so that the rotation and boost in the polar decomposition commute.

Proposition 4.23.

(i) *For each $e_0 \in M_1^+$ the set $\mathbf{G}_L^{e_0}$ is an open subset of $\dot{\mathbf{G}}_L$.*

(ii) $\bigcup_{e_0 \in M_1^+} \mathbf{G}_L^{e_0} = \dot{\mathbf{G}}_L$.

(iii) *$\dot{\mathbf{G}}_L$ is a two-sheeted covering space of $L_0 \setminus \{1\}$ when endowed with the covering map $\tilde{\lambda}$.*

Proof. If a sequence $\mu_n \to \mu$ in L_0 with $\hat{\rho}(\mu_n)\hat{\beta}(\mu_n) = \hat{\beta}(\mu_n)\hat{\rho}(\mu_n)$, then $\hat{\rho}(\mu)\hat{\beta}(\mu) = \hat{\beta}(\mu)\hat{\rho}(\mu)$. Namely, one has $\hat{\beta}(\mu_n)^{-1}\hat{\rho}(\mu_n)\hat{\beta}(\mu_n)\hat{\rho}(\mu_n)^{-1} = 1$ for all n, so $\hat{\beta}(\mu)^{-1}\hat{\rho}(\mu)\hat{\beta}(\mu)\rho^{-1}(\mu) = 1$ follows by continuity of the functions $\hat{\beta}$, $\hat{\rho}$, $\hat{\beta}(\cdot)^{-1}$ and $\hat{\rho}(\cdot)^{-1}$ and of the group product.

As a consequence, the set $L_0^{e_0}$ has a closed complement and hence is an open subset of L_0. Accordingly, $\mathbf{G}_L^{e_0} = \tilde{\lambda}^{-1}(L_0^{e_0})$ is open by continuity of $\tilde{\lambda}$. This proves (i).

It follows from Lemma 4.16 that $\bigcup_{e_0 \in M_1^+} L_0^{e_0} = L_0 \setminus \{1\}$, and this proves statement (ii) by continuity of $\tilde{\lambda}$.

By statements (i) and (ii), there is for each $g \in \dot{\mathbf{G}}_L$ an open neighbourhood restricted to which $\tilde{\lambda}$ is one-to-one and open. This proves (iii). □

So far we have established that $\dot{\mathbf{G}}_L^{e_0}$ is a Hausdorff space and that $\tilde{\lambda}$, restricted to $\dot{\mathbf{G}}_L^{e_0}$ is a two-sheeted covering map onto ${}_0 \setminus \{1\}$. For the remaining elements $\pi(a, a)$ and $\pi(a, -1)$ this will be shown now. This completes the necessary preparations for the proof of Theorem 4.1.

Proposition 4.24.

(i) $\mathbf{G}_L$ *is a Hausdorff space.*

(ii) $\tilde{\lambda}$ *is open.*

Proof of (i). Being a union of Hausdorff spaces, $\dot{\mathbf{G}}_L$ is a Hausdorff space, so it remains to prove that for each g there are disjoint neighbourhoods U_1 and U_g of 1 and $g \neq 1$, respectively, which implies that there are disjoint neighbourhoods $-U_1$ and $-U_g$ of -1 and $-g$.

$g \neq 1$ implies that $(\tilde{\alpha}(g), \tilde{\chi}(g)) \neq (0, 0)$. Since $\tilde{\alpha}$ and $\tilde{\chi}$ are continuous[1] and $(\tilde{\alpha}(h), \tilde{\chi}(h)) = (0, 0)$ implies $h = 1$, the open sets

$$\begin{array}{rccccc} U_1 := (\tilde{\alpha} \times \tilde{\chi})^{-1}(& [0, \varepsilon) & \times & [0, \varepsilon) &) \\ \text{and } U_g := (\tilde{\alpha} \times \tilde{\chi})^{-1}(& (\tilde{\alpha}(g) - \varepsilon, \tilde{\alpha}(g) + \varepsilon) & \times & (\tilde{\chi}(g) - \varepsilon, \tilde{\chi}(g) + \varepsilon) &) \end{array}$$

are disjoint for $\varepsilon > 0$ sufficiently small.

Proof of (ii). It has been shown that $\dot{\mathbf{G}}_L$ is a two-sheeted covering space when endowed with the covering map $\tilde{\lambda}$. Since $\tilde{\lambda}$ is continuous on all of $\mathbf{G}_L$, it remains to be shown that $\tilde{\lambda}$ is open at ± 1. L_0 is first countable, so it suffices to establish that for each sequence $\mu_n \to 1$ in L_0 there exists a sequence $g_n \to 1$ in $\mathbf{G}_L$ with $\tilde{\lambda}(g_n) = \mu_n$; note that the sequence $(-g_n)_n$ tends to -1 in this case.[2] For each n there is a $g_n \in \tilde{\lambda}^{-1}(\mu_n)$ with $\tilde{\alpha}(g_n) \leq \pi/2$. For any $\varepsilon > 0$ almost all g_n satisfy $(\tilde{\alpha}(g_n), \tilde{\chi}(g_n)) \in [0, \pi] \times [0, \varepsilon]$. Since this is a compact set, the sequence $(\tilde{\alpha}(g_n), \tilde{\chi}(g_n))$

[1]with respect to the *relative* topologies of the *closed* topological subspaces $[0, \pi]$ and $\mathbb{R}^{\geq 0}$ of $\mathbb{R}$.

[2]It suffices to consider sequences, since L_0 is first-countable (which we have not yet proved for $\mathbf{G}_L$ at this stage). Namely, let $U_g \subset \mathbf{G}_L$ be a neighbourhood of any $g \in \mathbf{G}_L$ and let $(\mu_n)_n$ be a sequence in L_0 converging to $\tilde{\lambda}(g)$. By assumption there is a sequence $g_n \to g$ with $\tilde{\lambda}(g_n) = \mu_n$. Since $g_n \to g$ and since U_g is a neighbourhood of g, one has $g_n \in U_g$ for almost all n, so $\mu_n = \tilde{\lambda}(g_n) \in \tilde{\lambda}(U_g)$ for almost all n. Since this holds for all sequences $\mu_n \to \tilde{\lambda}(g)$, one concludes that $\tilde{\lambda}(U_g)$ is a neighbourhood of $\tilde{\lambda}(g)$ in L_0 by first-countability.

has at least one accumulation point. $\tilde{\beta}(g_n)$ tends to 1 and $\tilde{\chi}(g_n)$ tends to zero, so all accumulation points are in $[0, \pi] \times \{0\}$.

The assumption $\mu_n \to 1$ further reduces the set of possible points to the set $\{(0, 0), (\pi, 0)\}$, and opting for $\tilde{\alpha}(g_n) \leq \pi/2$ rules out $(\pi, 0)$. So both $\tilde{\alpha}(g_n)$ and $\tilde{\chi}(g_n)$ tend to zero. It follows that g_n tends to 1. □

Now we are in the position to prove the remaining statements made in Sect. 4.2.1. Collecting the results, one obtains

Theorem 4.1.(i). $\mathbf{G}_L$ *is a two-sheeted covering space of* L_0 *when endowed with the covering map* $\tilde{\lambda}$.

Proof. $\dot{\mathbf{G}}_L$ is a cover of $L_0 \setminus \{1\}$ when endowed with the covering map $\tilde{\lambda}$, so all that remains to be shown is that $\tilde{\lambda}$ is a homeomorphism from some neighbourhood U of 1 or -1 onto $\tilde{\lambda}(U)$.

Since $\mathbf{G}_L$ is a Hausdorff space, there exist disjoint neighbourhoods $U_\pm$ of ± 1. Since $\tilde{\lambda}$ is open, the images $V_\pm := \tilde{\lambda}(U_\pm)$ are open. The intersection $V := V_+ \cap V_-$ is an open neighbourhood of $1 \in L_0$, and by continuity of $\tilde{\lambda}$, the sets $W_\pm := U \cap \tilde{\lambda}^{-1}(V_+ \cap V_-)$ are open neighbourhoods of $\pm 1 \in \mathbf{G}_L$, respectively. Since $W_\pm$ have been constructed in such a way that $\tilde{\lambda}(W_+) = U = \tilde{\lambda}(W_-)$, the restrictions $\tilde{\lambda}_\pm$ to $W_\pm$ are one-to-one and onto, and, since $\tilde{\lambda}$ is open, the inverse mappings are continuous. □

Theorem 4.1.(ii). $\mathbf{G}_L$ *is simply connected.*

Proof. $\bar{Z}$ is pathwise connected, so $\mathbf{M}_L = \bar{Z} \times \bar{Z}$ is pathwise connected, and, since π is continuous, $\mathbf{G}_L = \pi(\mathbf{M}_L)$ is pathwise connected. Since $\mathbf{G}_L$ is a two-sheeted covering group of L_0, and the fundamental group of L_0 is isomorphic to $\mathbb{Z}_2$, one concludes that $\mathbf{G}_L$ is homeomorphic to the universal cover of L_0. □

Theorem 4.1.(iii). *There is a unique group product* $\odot$ *on* $\mathbf{G}_L$ *with the property that the diagram*

$$
\begin{array}{ccc}
\mathbf{M}_L \times \mathbf{M}_L & \xrightarrow{\ \circ\ } & \mathbf{M}_L \\
\downarrow{\scriptstyle \pi\times\pi} & & \downarrow{\scriptstyle \pi} \\
\mathbf{G}_L \times \mathbf{G}_L & \xrightarrow{\ \odot\ } & \mathbf{G}_L \\
\downarrow{\scriptstyle \tilde{\lambda}\times\tilde{\lambda}} & & \downarrow{\scriptstyle \tilde{\lambda}} \\
L_1 \times L_1 & \xrightarrow{\ \cdot\ } & L_1
\end{array}
\tag{4.11}
$$

commutes.

Proof. The outer arrows of the diagram commute, so it suffices to prove existence and uniqueness of a group product conforming with the lower part. But it is well known that each simply connected covering space $\tilde{G}$ of a topological group G can be endowed with a unique group product $\odot$ such that G is a covering group.[3] □

[3]See, e.g., Propositions 5 and 6 in [Che46, Section I.VIII.].

From now on, the group product of $g, h \in \mathbf{G}_L$ will simply be denoted by gh instead of $g \odot h$. The last remaining proof was the one of Lemma 4.7, which demonstrates an important property of the group product in $\mathbf{G}_L$.

Lemma 4.7. *Given $h \in \mathbf{G}_L$ and $(c,d) \in \mathbf{M}_L$, one has*

$$h\pi(c,d)h^{-1} = \pi\left(\tilde{\lambda}(h)c, \tilde{\lambda}(h)d\right). \tag{4.12}$$

Proof. The function $\Gamma : \mathbf{G}_L \to \mathbf{G}_L$ defined by

$$\Gamma(h) := \pi\left(\tilde{\lambda}(h)c, \tilde{\lambda}(h)d\right)^{-1} h\pi(c,d)h^{-1}$$

has the property that $\tilde{\lambda}(\Gamma(h)) = 1$ and hence it takes values in the discrete set $\{\pm 1\} \subset \mathbf{G}_L$. Since Γ is continuous and L_0 is connected, Γ is constant, and, because $\Gamma(1) = 1$, it follows that $\Gamma(h) = 1$ for all h. □

4.2.3 The Subgroup $Spin(\mathbb{R}^{1+3})_0$ of the Spin Group $Spin(\mathbb{R}^{1+3})$

The group $\mathbf{G}_L$ introduced in the previous section characterises the universal covering group of $L_+^\uparrow$ by pairs of wedges. It is built from scratch and in its construction, in spite of the simple geometrical idea, technical obstructions have to be overcome. In the case of the rotation group similar problems occurred, and the model $Spin(\mathbb{R}^3)_0$ for the universal covering group provided a more elegant and simple tool. It is based on vectors and reflections in their orthogonal complement, but the connection to the class of wedges $\mathcal{W}$ and their modular conjugations is straightforward.

The subgroup $Spin(\mathbb{R}^{1+3})_0$ of $Spin(\mathbb{R}^{1+3})$, which is the realisation of the universal cover of the Lorentz group $L_+^\uparrow$ introduced in Section 2.4.1, is an elegant alternative to $\mathbf{G}_L$. Recall that the spin group $Spin(\mathbb{R}^{1+3})_0$ is a subgroup of the Clifford algebra $Cl(\mathbb{R}^{1+3})$. The Clifford algebra can be thought of as generated by the vectors in Minkowski space subject to the relations

$$x \cdot y + y \cdot x = 2g(x,y)\mathbb{1} \qquad \text{for } x, y \in \mathbb{R}^{1+3}.$$

The vector space $\mathbb{R}^{1+3}$ is canonically embedded in $Cl(\mathbb{R}^{1+3})$, and for the product in $Cl(\mathbb{R}^{1+3})$ we simply write vw instead of $v \cdot w$. The spin group $Spin(\mathbb{R}^{1+3})_0$ is the subgroup of $Cl(\mathbb{R}^{1+3})$ generated by products of even numbers of unit vectors. The covering map $\widetilde{Ad}$ was defined in eq. (2.36) and has the property that $\widetilde{Ad}_v(x) = -vxv - j_v x$ for $x, v \in \mathbb{R}^{1+3} \cap Cl(\mathbb{R}^{1+3})$ with $g(v,v) \neq 0$.

But in contrast to the rotationally invariant case, now the situation does not so easily reduce to the case of reflections in the orthogonal complement of unit vectors. The description of the spin group $Spin(\mathbb{R}^{1+3})_0$ by pairs of wedges is slightly more involved. The lessons learned in the construction of $\mathbf{G}_L$ help to analyse the geometrical structure of $Spin(\mathbb{R}^{1+3})_0$ and are especially useful for the definition of the representation in terms of modular conjugations. As in the rotationally invariant case the starting point is the geometric property of $\widetilde{Ad}_v$ being a reflection in the hyperplane orthogonal to v if $g(v,v) \neq 0$ (see eq. (2.9)). For a wedge a characterised by the zweibein ξ we have

$$j_a = j_\xi = j_{t_\xi} j_{x_\xi} = \widetilde{Ad}_{t_\xi x_\xi}.$$

Now we want to identify a subset of $Spin(\mathbb{R}^{1+3})_0$ characterising the wedges. Recall that $\mathbb{R}^{1+3}$ is identified with its canonical embedding in $Cl(\mathbb{R}^{1+3})$ and that the product in $Cl(\mathbb{R}^{1+3})$ is denoted by "$\cdot$". Consider the set $\underline{\mathcal{W}} := \{t_\xi \cdot x_\xi;\, \xi \in Z\}$. Note that $\underline{\mathcal{W}} = \{t_\xi \cdot x_\xi;\, \xi \in Z \text{ and } t_\xi \in V^+\}$, where V^+ denotes the forward light cone. These two characterisations of $\underline{\mathcal{W}}$ are closely related to the sets Z and Z^+ introduced in equations (2.3) and (4.7) and the equivalence relation $\tilde{\sim}$ defined on them. The following proposition shows that here the product in the spin group encodes the equivalence relation $\tilde{\sim}$, which identifies zweibeine describing the same wedge. If t is a timelike vector, then define

$$t^+ = \begin{cases} t, & \text{if } t \in V^+, \\ -t & \text{else.} \end{cases} \tag{4.13}$$

The following proposition shows that the map

$$l : \mathcal{W} \to \underline{\mathcal{W}}; \quad W_\xi \mapsto t_\xi^+ \cdot x_\xi \tag{4.14}$$

is well-defined.

Proposition 4.25. *Let $\xi, \chi \in Z^+$. Then $W_\xi = W_\chi$ if and only if $t_\xi \cdot x_\xi = t_\chi \cdot x_\xi$ in the spin group $Spin(\mathbb{R}^{1+3})_0$. Therefore l is well-defined and bijective.*

Proof. Let ξ and χ in Z characterise the same wedge $a := W_\xi = W_\chi$. We have to show that $t_\xi^+ \cdot x_\xi = t_\chi^+ \cdot x_\chi$. Since the stabiliser group of a wedge is an abelian two-parameter group, there exists a one-parameter group of Lorentz transformations γ in the stabiliser with $\gamma(0) = 1$ and $\gamma(1)\xi = \chi$. The function $f(t) = (\gamma(t)t_\xi)\cdot(\gamma(t)x_\xi)\cdot(t_\xi \cdot x_\xi)^{-1}$ with $t \in [0,1]$ can only take the values $\pm\mathbb{1}$ because $\widetilde{Ad}_{f(t)} = 1$:

$$\widetilde{Ad}_{f(t)} = j_{\gamma(t)t_\xi} j_{\gamma(t)x_\xi} j_{t_\xi} j_{x_\xi} = \gamma(t) j_\xi \gamma(t)^{-1} j_\xi = j_\xi j_\xi = 1.$$

Since $f(0) = \mathbb{1}$ and f is continuous, we have $f(t) = \mathbb{1}$ for $t \in [0,1]$ and l is well-defined.

l is obviously surjective. It is also injective: Let ξ and β be two zweibeine with $l(\xi) = l(\chi)$, i.e. $t_\xi^+ \cdot x_\xi = t_\chi^+ \cdot x_\chi$. This implies $j_\xi = \widetilde{Ad}(t_\xi^+ \cdot x_\xi) = \widetilde{Ad}(t_\chi^+ \cdot x_\chi) = j_\chi$, so the edges of W_ξ and W_χ coincide. This leaves the possibilities $W_\xi = W_\chi$ and $W_\xi = W_{-\chi}$. But $W_\xi = W_{-\chi}$ leads to a contradiction, implying $l(\xi) = l(-\chi) = t_\chi^+ \cdot (-x_\chi) = -t_\chi^+ \cdot x_\chi$, which by assumption should equal $l(\chi) = t_\chi^+ \cdot x_\chi$. The second statement follows by definition of Z^+. □

Via the bijection l it is justified to use the term "wedge" also for elements of the set $\underline{\mathcal{W}} = \{t_\xi \cdot x_\xi;\ \xi \in Z\}$. This admits a useful notation for the wedges. For two wedges a and b defined by the zweibeine ξ and χ, respectively, we set $\xi \cdot \chi := t_\xi^+ \cdot x_\xi \cdot t_\chi^+ \cdot x_\chi$ and $a \cdot b := \xi \cdot \chi$. Frequently, as an even shorter notation ab is used instead of $a \cdot b$.

Lemma 4.26. *With the notation introduced above for the zweibeine and wedges one finds*

$$Spin(\mathbb{R}^{1+3})_0 = \{a \cdot b;\ a, b \in \mathcal{W}\} =: \mathcal{W} \cdot \mathcal{W}.$$

Proof. The product $a \cdot b$ of two wedges is contained in $Spin(\mathbb{R}^{1+3})_0$. Conversely, every restricted Lorentz transformation can be written as the product of two reflections in edges of wedges a and b. Choosing zweibeine ξ and χ for these wedges with $a = W_\xi$ and $b = W_\chi$, one has $j_\xi j_\chi = j_{t_\xi} j_{x_\xi} j_{t_\chi} j_{x_\chi}$. The pre-images of $j_\xi j_\chi$ under $\widetilde{Ad}$ are $\pm(t_\xi x_\xi)(t_\chi x_\chi) = \pm a \cdot b$, and both are elements of $\mathcal{W} \cdot \mathcal{W}$. So we proved that the fibre in $Spin(\mathbb{R}^{1+3})_0$ over every Lorentz transformation contains two elements in $\mathcal{W} \cdot \mathcal{W}$. □

4.2.4 Structure of $Spin(\mathbb{R}^{1+3})_0$ and Isomorphism with G_L

Since $Spin(\mathbb{R}^{1+3})_0$ and $\mathbf{G}_L$ are universal covers of $L_+^\uparrow$, they have to be isomorphic. To establish this isomorphism we will need some observations of properties inherent in the structure of $Spin(\mathbb{R}^{1+3})_0$ if described as $\mathcal{W} \cdot \mathcal{W}$. Let $a, b, c, d \in \mathcal{W}$. The short notation ab instead of $a \cdot b$ introduced in the preceding section will be employed for $a, b \in \mathcal{W}$. First of all note that $a^2 = \mathbb{1}$ because, if $\xi \in Z$ with $a = t_\xi^+ x_\xi$, then $a^2 = t_\xi^+ x_\xi t_\xi^+ x_\xi = -(t_\xi^+ t_\xi^+)(x_\xi x_\xi) = -\mathbb{1}(-\mathbb{1}) = \mathbb{1}$. In Lemma 3.4 it was shown that for $a, b \in \mathbb{R}^3 \subset Spin(\mathbb{R}^3)_0$

$$aba = -j_a b$$

holds. This is also true for vectors in $Spin(\mathbb{R}^3)_0$ and for $a, b \in \mathcal{W} \subset Spin(\mathbb{R}^{1+3})$. Choose $\xi, \chi \in Z^+$ with $a = t_\xi x_\xi$ and $b = t_\chi x_\chi$, then

$$\begin{aligned} aba &= t_\xi x_\xi (t_\chi x_\chi) t_\xi x_\xi \\ &= -x_\xi t_\xi (t_\chi x_\chi) t_\xi x_\xi \\ &= -(j_{x_\xi} j_{t_\xi} t_\chi)(j_{x_\xi} j_{t_\xi} x_\chi) \\ &= -(j_\xi t_\chi)(j_\xi x_\chi) = -j_a b. \end{aligned}$$

This enters in the proof of the following lemma.

Lemma 4.27. *If* $(ab)^2 = \mathbb{1}$ *then* $a = \pm b$.

Proof. $\mathbb{1} = abab = -(j_a b)b$ implies $-b = -b^{-1} = j_a b$. Since the only reflection mapping b to $-b$ is j_b, one has $j_a = j_b$ and therefore $a = \pm b$ □

For convenience, the definition of $\mathbf{G}_L$ is recalled. As mentioned before, any restricted Lorentz transformation can be written as the product of two reflections in edges of wedges. For pairs of wedges we define an equivalence relation: Let (a, b) and (c, d) be pairs of wedges. Set $(a, b) \sim (c, d)$ if the Lorentz transformations $j_a j_b$ and $j_c j_d$ coincide and if there exists a (restricted) Lorentz transformation μ commuting with $j_a j_b$ and satisfying $(\mu^2 a, \mu^2 b) = (c, d)$ or $(\mu^2 a, \mu^2 b) = (-c, -d)$.

Let $\mathbf{G}_L$ be the quotient space $\mathbf{G}_L := \mathcal{W} \times \mathcal{W} / \sim$ and π the corresponding projection map. In Section 4.2.2 (cf. [KL07]) we proved that $\mathbf{G}_L$ is the universal cover of the restricted Lorentz group. In the following the covering map from $\mathbf{G}_L$ is denoted by $\tilde{\lambda}_L$, and for the covering map from $Spin(\mathbb{R}^{1+3})_0$ the shorthand $\tilde{\lambda} := \widetilde{Ad}$ is used. The following theorem establishes the isomorphism between $\mathbf{G}_L$ and $Spin(\mathbb{R}^{1+3})_0 = \mathcal{W} \cdot \mathcal{W}$.

Theorem 4.2 (Isomorphism of $\mathbf{G}_L$ and $Spin(\mathbb{R}^{1+3})_0$). *The map*

$$\iota : \mathbf{G}_L \to Spin(\mathbb{R}^{1+3})_0; \quad g = \pi(a, b) \mapsto ab$$

is well-defined and an isomorphism between $\mathbf{G}_L$ and $Spin(\mathbb{R}^{1+3})_0$.

Proof. Let $a, b, c, d \in \mathcal{W}$. First we will show that $(a, b) \sim (c, d)$ if and only if $ab = cd$, proving ι to be well-defined and injective. That it is surjective has been shown in Lemma 4.26. So assume $(a, b) \sim (c, d)$. Using the shorthand $j_a j_b =: \mu$, by definition of the equivalence relation, there exists a Lorentz transformation $\nu \in L_+^\uparrow$ with $[\mu, \nu] = 1$ and $\nu^2(a, b) = \pm(c, d)$. Now choose $h \in \mathcal{W} \cdot \mathcal{W}$ with $\tilde{\lambda}(h) = \nu$. The sign in $\pm(c, d)$ can be neglected since $cd = (-c)(-d)$ and one has $cd = (\nu^2 a)(\nu^2 b) = (\tilde{\lambda}(h)^2 a)(\tilde{\lambda}(h)^2 b) = h^2 (ab) h^{-2} = ab$. The last two equalities follow from Lemmas 3.4 and 3.5 together with $[\mu, \nu] = 1$. Both lemmas are valid in the present setting. Conversely, let $ab = cd$. Since $\tilde{\lambda}(ab) = \tilde{\lambda}(cd)$, Proposition 4.4 (*iii*) implies either $(a, b) \sim (c, d)$ or $(a, b) \sim (c, -d)$. But $(a, b) \sim (c, -d)$ leads to the contradiction $cd = c(-d) = -cd$. So the diagram

$$\begin{array}{ccc} \mathbf{G}_L & \underset{\iota^{-1}}{\overset{\iota}{\rightleftarrows}} & Spin(\mathbb{R}^{1+3})_0 \\ & {\scriptstyle \tilde{\lambda}_L} \searrow \quad \swarrow {\scriptstyle \tilde{\lambda}} & \\ & L_+^\uparrow & \end{array}$$

commutes, where $\tilde{\lambda}_L$ and $\tilde{\lambda}$ denote the covering maps. Since these are homomorphisms and local homeomorphisms, ι and ι^{-1} are homomorphisms. □

Independent of the structure of $\mathbf{G}_L$ the preceding theorem establishes for wedges $a, b, c, d \in \mathcal{W}$ an interesting fact. If $ab = cd$ then there exists a Lorentz transformation μ which commutes with $\tilde{\lambda}(ab)$ and satisfies $\mu^2(a, b) = \pm(c, d)$. This is just the condition for (a, b) and (c, d) to be equivalent.

4.3 Construction of the Representation

So far the analysis was concerned with the universal covering group of the restricted Lorentz group. This was motivated by the question how the product of pairs of modular conjugation behaves, but the results are independent of this connection to modular theory and quantum field theory and are interesting on their own. In this section a representation of the covering group of the Lorentz group will be defined. The characterisation of the covering group by pairs of wedges and its geometrical features are essential to show that for an element of the covering group the product of two modular conjugations $J_a J_b$ is independent of the pair of wedges (a, b) used to specify it.

For the covering maps from $Spin(\mathbb{R}^{1+3})_0$ and $\mathbf{G}_L$ we use the symbols $\tilde{\lambda} := \widetilde{Ad}$ and $\tilde{\lambda}_L$, respectively. Let the timelike unit vector e_0 single out a time direction. We will construct a representation of $Spin(\mathbb{R}^{1+3})_0$, although all statements can be translated to $\mathbf{G}_L$ as the isomorphism in Theorem 4.2 suggests. One simply has to replace some expressions and translate $ab = cd$ to $\pi(a, b) = \pi(c, d)$ or $(a, b) \sim (c, d)$.

Note that everything known about the decomposition of a Lorentz transformation into pairs of reflections in edges of wedges can be lifted to $\mathcal{W} \times \mathcal{W}$ up to a sign. For instance, by Lemma 4.2 (cf. [KL07, Lemma 4]) there exists for a Lorentz transformation $j_a j_b$ an element $\mu \in L_+^\uparrow$ with $\mu^2 = j_a j_b$ and $a = \pm\mu b$. This implies $g = \pm(\mu b) \cdot b$ for $g \in Spin(\mathbb{R}^{1+3})_0$ with $\tilde{\lambda}(g) = j_a j_b$. Another example is presented in the following lemma, which is the counterpart of Lemma 4.19.

Lemma 4.28. *Let $g \in Spin(\mathbb{R}^{1+3})_0$ and let $\rho\beta$ be the unique rotation-boost decomposition of $\tilde{\lambda}(g)$. Choose a unit vector $e \in \mathbb{R}^{1+3} \cap e_0^\perp$ orthogonal to the rotation axis of ρ (if defined) and orthogonal to the boost direction of β (if defined). Set $\bar{e} := W_{(e_0,e)}$. Then g can be written in the form $g = (\tau\bar{e})(\beta^{-1/2}\bar{e})$ with a rotation τ satisfying $\tau^2 = \rho$. If $g = \mathbb{1}$ then $\tau\bar{e} = \bar{e}$, and if $g = -\mathbb{1}$ then $\tau\bar{e} = -\bar{e}$. In the case of ρ being nontrivial, τ is uniquely determined by the condition that the axes of τ and ρ shall coincide.*

Proof. The rotation-boost decomposition $\rho\beta$ of $\tilde{\lambda}(g)$ can be written as $\rho\beta = j_{\tau\bar{e}} j_{\beta^{-1/2}\bar{e}}$ with $\bar{e} = W_{(e_0,e)} \in \mathcal{W}$ and rotation τ as described above. So $g = (\pm\tau\bar{e})(\beta^{-1/2}\bar{e})$ for one of the signs, τ being one of the two square roots of ρ. In the case of nontrivial ρ its square root τ is uniquely fixed by choosing the positive sign. □

Some basic, but essential properties of the modular conjugations will be recalled from Section 3.3 (see equations (3.6), (3.7) and (3.8)). k is the statistics operator and $\kappa = (1+ik)/(1+i)$. The following holds:

$$\begin{aligned} kJ_a k &= J_a, \\ J_a \kappa &= \kappa^\dagger J_a, \\ \kappa^\dagger J_a \kappa &= J_{-a}, \\ J_a J_b J_a &= J_{-j_a b}. \end{aligned}$$

These consequences of Proposition 2.6 and Lemma 3.6 will be used extensively in what follows without further mentioning. Now the goal in this section is to prove that the $\tilde{W}(g) := J_a J_b$ for $g = ab$ is well-defined, independent of the choice of e_0 and is a representation. An important result in this direction is the following theorem.

Theorem 4.3. *Let for $g \in Spin(\mathbb{R}^{1+3})_0$ $g = (\tau\bar{e})(\beta^{-1/2}\bar{e})$ be the decomposition specified in the previous Lemma 4.28. The map $\tilde{W} : Spin(\mathbb{R}^{1+3})_0 \to B(\mathcal{H})$ given by*

$$\tilde{W}(g) = \tilde{W}((\tau\bar{e})(\beta^{-1/2}\bar{e})) = J_{\tau\bar{e}} J_{\beta^{-1/2}\bar{e}}$$

is well-defined.

Proof. If $g \in Spin(\mathbb{R}^{1+3})_0$ is trivial, then $g = aa$ for any wedge $a \in \mathcal{W}$. Since J_a is an involution, $\tilde{W}(\mathbb{1}) = J_a J_a = \mathbb{1}$ independent of $a \in \mathcal{W}$. If $g = -\mathbb{1}$, then $g = -aa$ for any wedge a. The property

$$J_{-a} = \kappa^\dagger J_a \kappa = J_a \kappa^2 = J_a k$$

implies $J_{-a} J_a = k$ independent of a and one can set $\tilde{W}(-\mathbb{1}) = J_a J_{-a} = k$. Recall that the polar decomposition and the notion of rotation and boost depend on the choice of the time direction specified above via $e_0 \in M_1^+$.

For each $r \in Spin(\mathbb{R}^{1+3})_0$ with $\tilde{\lambda}(r) \in R - \{1\}$ there exists a unique rotation τ with $\tau^2 = \tilde{\lambda}(r)$ and $r = (\tau\bar{e})(\bar{e})$ for all time-zero unit vectors $e \in \mathrm{FP}(\lambda(r))^\perp$. Choose e accordingly. For each pair $(\bar{e}', \bar{f}')$ with $r = \bar{e}'\bar{f}'$, there exists a rotation ρ with $\rho\lambda(\tau\bar{e}, \bar{e})\rho^{-1} = \lambda(\tau\bar{e}, \bar{e})$ and $\rho^2(\tau\bar{e}, \bar{e}) = (\bar{e}', \bar{f}')$. Because $a \mapsto J_a$ and, hence, also the map W is continuous by assumption of modular $\mathrm{P}_1\mathrm{CT}$-symmetry, one can mimic the proof of in Ref. [BS04, Lemma 2.4] in order to show that $W(\tau\bar{e}, \bar{e}) = W(\bar{e}', \bar{f}')$, and one can define a unitary operator by $\tilde{W}(r) := W(\tau\bar{e}, \bar{e})$.

Each $b \in Spin(\mathbb{R}^{1+3})_0$ with $\tilde{\lambda}(b) \in B$ is generated by pairs of wedges either of the form $(\bar{e}, \beta\bar{e})$ or $(\bar{e}, -\beta\bar{e})$ with $e \in S^2$, where $\beta := \tilde{\lambda}(b)^{-1/2}$. The one-parameter group of rotations about the boost direction of β acts transitively on the set of such elements. If $(\bar{e}, \pm\beta\bar{e})$ and $(\bar{f}, \pm\beta\bar{f})$ are two decompositions of b, one can again use the reasoning of Ref. [BS04] in order to show that $W(\bar{f}, \pm\beta\bar{f}) = W(\bar{e}, \pm\beta\bar{e})$, and one can define $\tilde{W}(b) := J_{\bar{e}} J_{\pm\beta\bar{e}}$.

The polar decomposition in the Lorentz group can be lifted to a polar decomposition in $Spin(\mathbb{R}^{1+3})_0$. Namely, given an arbitrary $g \in Spin(\mathbb{R}^{1+3})_0$, there exist

$r_g, b_g \in Spin(\mathbb{R}^{1+3})_0$ with $\tilde{\lambda}(r_g) \in R$, $\tilde{\lambda}(b_g) \in B$ and $r_g b_g = g$. This decomposition is unique up to replacing r_g and b_g by $-r_g$ and $-b_g$, respectively. It is compatible with the representation of g as $g = (\tau\bar{e})(\beta^{-1/2}\bar{e}) = (\tau\bar{e})\bar{e}\bar{e}(\beta^{-1/2}\bar{e})$ for $e \in S^2$ orthogonal to the rotation axis of $\tilde{\lambda}((\tau\bar{e})\bar{e})$ and to the boost direction of $\tilde{\lambda}(\bar{e}(\beta^{-1/2}\bar{e}))$ (Lemma 4.28). We can set $r_g := (\tau\bar{e})\bar{e}$ and $b_g := \bar{e}(\beta^{-1/2}\bar{e})$.

The operator $\tilde{W}(g) := \tilde{W}(r_g)\tilde{W}(b_g) = J_{\tau\bar{e}}J_{\bar{e}}J_{\bar{e}}J_{\beta^{-1/2}\bar{e}} = J_{\tau\bar{e}}J_{\beta^{-1/2}\bar{e}}$ does not depend on the choice of this polar decomposition. □

In the proof of $\tilde{W}$ being well-defined by adaption of [BS04, Lemma 2.4], the continuity of the map $a \mapsto J_a$ enters. Note that $\tilde{W}(g)$ depends on e_0 as it stands because of the polar decomposition involved in its definition. Nevertheless the index will be dropped for the time being. This will be justified by the arguments following the proof of the subsequent theorem.

Theorem 4.4.

(i) $\tilde{W}$ *is a representation.*

(ii) There is a representation $\tilde{D}$ *of* $Spin(\mathbb{R}^{1+3})_0$ *in* $\mathfrak{C}$ *such that*

$$\tilde{W}(g)F(\mathfrak{c}, f)\tilde{W}(g)^\dagger = F(\tilde{D}(g)\mathfrak{c}, \tilde{\lambda}(g)f) \quad \text{for all } g, \mathfrak{c}, f, \tag{4.15}$$

where $\tilde{\lambda}(g)f := f(\tilde{\lambda}(g)^{-1}\,\cdot)$.

Proof of (i). We will prove the representation property of $\tilde{W}$ step by step. It has been shown in Ref. [Kuc05] (see Chapter 3 here) that $\tilde{W}$ is a representation of the subgroup $Spin(\mathbb{R}^3)_0 = \tilde{\lambda}^{-1}(R)$ of $Spin(\mathbb{R}^{1+3})_0$.

Lemma 4.29. *Consider* $b, r \in Spin(\mathbb{R}^{1+3})_0$ *with* $\tilde{\lambda}(b) \in B$ *and* $\tilde{\lambda}(h) \in R$. *Then*

$$\tilde{W}(r)\tilde{W}(b)\tilde{W}(r)^\dagger = \tilde{W}\left(rbr^{-1}\right).$$

Proof. Let $e \in S^2$ be orthogonal to the rotation axis of $\tilde{\lambda}(r)$ and to the boost direction of $\beta := \tilde{\lambda}(b)$. Then $b = \bar{e}(\beta^{-1/2}\bar{e})$ and $\tilde{W}(b) = J_{\bar{e}}J_{\beta^{-1/2}\bar{e}}$. Note that

$$\tilde{\lambda}(r)\beta^{-1/2}\bar{e} = \tilde{\lambda}(r)\beta^{-1/2}W_{(e_0,e)} = W_{(\beta^{-1/2}e_0, \tilde{\lambda}(r)e)} = \beta^{-1/2}\tilde{\lambda}(r)W_{(e_0,e)} = \beta^{-1/2}\tilde{\lambda}(r)\bar{e}.$$

Now it follows from eq. (3.8) and Lemma 4.7 that

$$\begin{aligned}\tilde{W}(r)\tilde{W}(\bar{e}(\beta^{-1/2}\bar{e}))\tilde{W}(r)^\dagger &= \tilde{W}(r)J_{\bar{e}}J_{\beta^{-1/2}\bar{e}}\tilde{W}(r)^\dagger = J_{\tilde{\lambda}(r)\bar{e}}J_{\tilde{\lambda}(r)\beta^{-1/2}\bar{e}} \\ &= W(\tilde{\lambda}(r)\bar{e}, \beta^{-1/2}\tilde{\lambda}(r)\bar{e}) = \tilde{W}\left((\tilde{\lambda}(r)\bar{e})(\beta^{-1/2}\tilde{\lambda}(r)\bar{e})\right) \\ &= \tilde{W}(rbr^{-1}).\end{aligned}$$

□

If $(b_t)_t$ is a one-parameter subgroup of $Spin(\mathbb{R}^{1+3})_0$ with $\tilde{\lambda}(b_t) \in B$, it follows from the results of Ref. [BS04] that $\tilde{W}(b_s)\tilde{W}(b_t) = \tilde{W}(b_{s+t})$. This implies for b_1, b_2 with $\tilde{\lambda}(b_1)$ and $\tilde{\lambda}(b_2)$ being boosts in the same direction that $\tilde{W}(b_1)\tilde{W}(b_2) = \tilde{W}(b_1 b_2)$. The reason is that the stabiliser group of a wedge $\bar{e}$ is generated by the two one-parameter groups of boosts in the direction e and rotations about e. So there is a one parameter group of boosts connecting $\tilde{\lambda}(b_1)$ and $\tilde{\lambda}(b_2)$. The unique lift of this group connects $\pm b_1$ and $\pm b_2$ for one choice of signs. For example if it connects $-b_1$ and b_2, then

$$\tilde{W}(b_1)\tilde{W}(b_2) = k\tilde{W}(-b_1)\tilde{W}(b_2) = k\tilde{W}(-b_1 b_2) = \tilde{W}(b_1 b_2).$$

The other cases are similar. If $\tilde{\lambda}(b)$ and $\tilde{\lambda}(r)$ are in the stabiliser of some wedge $\bar{e}$ for $e \in S^2$, then $rbr^{-1} = r$ by Lemma 3.5 and its consequences. This implies

$$\tilde{W}(r)\tilde{W}(b) = \tilde{W}(r)\tilde{W}(b)\tilde{W}(r)^{-1}\tilde{W}(r) = \tilde{W}(rbr^{-1})\tilde{W}(r) = \tilde{W}(b)\tilde{W}(r).$$

The following lemma shows that in most cases the property $ab = cd$ implies $J_a J_b = J_c J_d$.

Lemma 4.30. *If $\tilde{\lambda}(g)$ is in the stabiliser of some wedge, then $W(a,b) = \tilde{W}(g)$ for all $a, b \in \mathcal{W}$ with $g = ab$.*

Proof. Without loss of generality, suppose that $\tilde{\lambda}(g)$ is in the stabiliser of $\bar{e}$ for some time-zero unit vector e. If $g = r_g b_g$ and $h = r_h b_h$ with $\tilde{\lambda}(h) \in \mathfrak{S}(\bar{e})$, then Lemma 4.29 implies

$$\begin{aligned} \tilde{W}(h)\tilde{W}(g)\tilde{W}(h)^\dagger &= \tilde{W}(r_h)\tilde{W}(b_h) \cdot \tilde{W}(r_g)\tilde{W}(b_g) \cdot \tilde{W}(b_h)^\dagger \tilde{W}(r_h)^\dagger \\ &= \tilde{W}(g). \end{aligned} \tag{4.16}$$

Let $(a,b) \in \mathcal{W} \times \mathcal{W}$ satisfy $J_a J_b = \tilde{W}(g)$. If $cd = ab$ and $(a,b) \neq (c,d)$, then there exists a $\mu \in L_+^\uparrow$ with $\mu^2 \neq 1$, commuting with $\tilde{\lambda}(g)$ and satisfying $\mu^2(a,b) = \pm(c,d)$. This is part of the isomorphism $\mathbf{G}_L \cong Spin(\mathbb{R}^{1+3})_0$ and an easy consequence of Proposition 4.4. Since $\mathfrak{S}(\bar{e})$ is a maximal abelian group and since $\tilde{\lambda}(g) \in \mathfrak{S}(\bar{e})$ by assumption, one concludes $\mu \in \mathfrak{S}(\bar{e})$, and for each h with $\tilde{\lambda}(h) = \mu^2$ one obtains from eq. (4.16)

$$W(c,d) = W(\pm\mu^2(a,b)) = W(\mu^2(a,b)) = \tilde{W}(h)\tilde{W}(g)\tilde{W}(h)^\dagger = \tilde{W}(g) = W(a,b). \quad \square$$

Proof of (i) (contd.).

To complete the proof of the representation property let $g, h \in Spin(\mathbb{R}^{1+3})_0$ be arbitrary with polar decomposition $r_g b_g$ and r_h, b_h. Then

$$\begin{aligned} \tilde{W}(g)\tilde{W}(h) &= \tilde{W}(r_g)\tilde{W}(b_g)\tilde{W}(r_h)\tilde{W}(b_h) \\ &= \tilde{W}(r_g)\tilde{W}(r_h)\left(\tilde{W}(r_h)^\dagger\tilde{W}(b_g)\tilde{W}(r_h)\right)\tilde{W}(b_h) \\ &=: \tilde{W}(r_g r_h)\,\tilde{W}(b_f)\tilde{W}(b_h). \end{aligned}$$

The last two terms implement the Lorentz transformation

$$\tilde{\lambda}(b_f)\tilde{\lambda}(b_h) = \tilde{\lambda}(b_f)^{1/2}\left(\tilde{\lambda}(b_f)^{1/2}\tilde{\lambda}(b_h)\tilde{\lambda}(b_f)^{1/2}\right)\tilde{\lambda}(b_f)^{-1/2},$$

which is conjugated to the boost in the brackets and therefore in the stabiliser of some wedge. So for a time-zero unit vector e orthogonal to the boost directions of $\tilde{\lambda}(b_f)$ and $\tilde{\lambda}(b_h)$ Lemma 4.30 yields

$$\begin{aligned}\tilde{W}(b_f b_h) &= J_{\pm\tilde{\lambda}(b_f)^{1/2}\bar{e}} J_{\tilde{\lambda}(b_h)^{-1/2}\bar{e}} \\ &= J_{\pm\tilde{\lambda}(b_f)^{1/2}\bar{e}} J_{\bar{e}}^2 J_{\tilde{\lambda}(b_h)^{-1/2}\bar{e}} = J_{\bar{e}} J_{\pm\tilde{\lambda}(b_f)^{-1/2}\bar{e}} J_{\bar{e}} J_{\tilde{\lambda}(b_h)^{-1/2}\bar{e}} \\ &= \tilde{W}(b_f)\tilde{W}(b_h)\end{aligned}$$

for one of the signs. Now write $b_f b_h =: d = r_d b_d$, then

$$\begin{aligned}\tilde{W}(g)\tilde{W}(h) &= \tilde{W}(r_g r_h r_d)\tilde{W}(b_d) \\ &= \tilde{W}(r_g r_h r_d b_d) \\ &= \tilde{W}(gh),\end{aligned}$$

which completes the proof.
Proof of (ii). Define a map D from $\mathcal{W}\times\mathcal{W}$ into the automorphism group $\mathrm{Aut}(\mathfrak{C})$ of $\mathfrak{C}$ by $D(a,b) := C_a C_b$. If $ab = cd$ and $\tilde{W}(ab) = J_a J_b = J_c J_d = \tilde{W}(cd)$, then modular $\mathrm{P_1CT}$-symmetry implies

$$\begin{aligned}F(C_a C_b \mathfrak{c}, j_a j_b f) &= J_a J_b F(\mathfrak{c}, f) J_b J_a \\ &= J_c J_d F(\mathfrak{c}, f) J_d J_c \\ &= F(C_c C_d \mathfrak{c}, j_c j_d f) \\ &= F(C_c C_d \mathfrak{c}, j_a j_b f)\end{aligned}$$

for all $\mathfrak{c}$ and all f. Using assumption (i) concerning redundancies in the component space (see on page 33), one obtains $C_a C_b \mathfrak{c} = C_c C_d \mathfrak{c}$ for all $\mathfrak{c}$, so $D(a,b) = D(c,d)$, and a map $\tilde{D} : Spin(\mathbb{R}^{1+3})_0 \to \mathrm{Aut}(\mathfrak{C})$ is defined by $\tilde{D}(ab) := D(a,b)$. This map $\tilde{D}$ now inherits the representation property from $\tilde{W}$. □

By the isomorphism $Spin(\mathbb{R}^{1+3})_0 \cong \mathbf{G}_L$ (Theorem 4.2), we can immediately compare the representation of $Spin(\mathbb{R}^{1+3})_0$ defined in Theorem 4.4 and the one of $\mathbf{G}_L$ defined in [KL07]. Let $g \in Spin(\mathbb{R}^{1+3})_0$, $\underline{g} \in \mathbf{G}_L$ with decomposition $g = (\tau\bar{e})(\beta^{-1/2}\bar{e})$ and $\underline{g} = \pi(\tau\bar{e}, \beta^{-1/2}\bar{e})$ as in Lemma 4.19 and Lemma 4.28, respectively. Then

$$\tilde{W}(g) = J_{\tau\bar{e}} J_{\beta^{-1/2}\bar{e}} = \tilde{W}(\pi(\tau\bar{e}, \beta^{-1/2}\bar{e})) = \tilde{W}(\underline{g})$$

with the right-hand side referring to the representation $\tilde{W}$ defined in [KL07].

That the construction of $\tilde{W}$ is indeed independent of the choice of e_0 can be seen most easily by employing the properties of the equivalence relation upon which $\mathbf{G}_L$ is based. Let $g \in \mathbf{G}_L$, $a, b \in \mathcal{W}$ with $\tilde{W}(g) = J_a J_b$ and let μ be a restricted Lorentz transformation commuting with $j_a j_b$. The set of squares of such Lorentz transformations acts transitively up to a sign on the class $\pi(a,b)$ because by definition of the equivalence relation $\pi(a,b) = \{\pm\mu^2(a,b);\ [\mu, j_a j_b] = 1\}$. Choose $h \in \tilde{\lambda}^{-1}(\mu)$, then, by Lemma 3.5, h^2 commutes with g. Taking $c, d \in \mathcal{W}$ with $\tilde{W}(h) = J_c J_d$, one computes

$$\begin{aligned} J_a J_b &= \tilde{W}(g) = \tilde{W}(h^2 g h^{-2}) = \tilde{W}(h)^2 \tilde{W}(g) \tilde{W}(h)^{-2} \\ &= (J_c J_d)^2 J_a J_b (J_d J_c)^2 = J_{\mu^2 a} J_{\mu^2 b} \\ &= J_{-\mu^2 a} J_{-\mu^2 b}. \end{aligned} \tag{4.17}$$

So for $g \in \mathbf{G}_L$ the product $J_a J_b$ does not depend on the choice of wedges a, b with $g = \pi(a,b)$. Therefore the choice of e_0 which was made to render the notion of polar composition meaningful does not affect the representation $\tilde{W}$. It is possible to simply define $\tilde{W}(g) := J_a J_b$ for any pair of wedges (a,b) with $\pi(a,b) = g$ (or stated in terms of the spin group, for any pair of wedges (a,b) with $g = ab$).

At this point it becomes again obvious that the geometrical properties inherent in the group $\mathbf{G}_L$ are also important for the seemingly more elegant formulation with the help of spin groups. The Propositions 4.5 and 4.6, as well as the lemmas in any case for establishing them, have to be proven nevertheless to gain the relevant structural results for the spin group.

The proof of the spin-statistics theorem proceeds exactly as the proof of Theorem 3.4 in Chapter 3. The fact that $\tilde{W}(-\mathbb{1}) = k$ stays unaltered and the statement coincides with the usual result for a irreducible representation $\tilde{D}$ with spin s. Integer spin fields obeying anticommutation relations have to vanish as well as half-integer spin fields with underlying commutation relations. The precise theorem will not be stated here, because it occurs once again in the next chapter, in which the representation is extended to the universal cover of the restricted Poincaré group.

5 Poincaré Invariance

In this chapter the representation of the universal cover of the Lorentz group will be extended by a representation of the translation group in Minkowski space. One obtains a representation of the double cover of the restricted Poincaré group in a straightforward way. This is the final step in the proof of the spin-statistics theorem from the assumption of modular $\mathrm{P}_1\mathrm{CT}$ symmetry for the full symmetry group of quantum field theory in four spacetime dimensions [Lor07].

The double cover of the Poincaré group is the semidirect product of the translation group $T \cong \mathbb{R}^{1+3}$ and the universal cover of the Lorentz group $\tilde{L}_+^\uparrow$. Let us recall the definition here. Denote by ϕ the map from $\tilde{L}_+^\uparrow \cong Spin(\mathbb{R}^{1+3})_0$ into the automorphism group of T given by

$$\phi(g)(x) = \tilde{\lambda}(g)x, \quad \text{where } g \in Spin(\mathbb{R}^{1+3})_0, x \in \mathbb{R}^{1+3} \cong T. \tag{5.1}$$

Then the semidirect product $T \ltimes_\phi Spin(\mathbb{R}^{1+3})_0$ is the set $T \times Spin(\mathbb{R}^{1+3})_0$ together with the multiplication law

$$(x, g)(y, h) = (x + \phi(g)(y), gh) = (x + \tilde{\lambda}(g)y, gh), \tag{5.2}$$

which is induced by ϕ. The equivalent structure $T \ltimes SL(2, C)$ is called *inhomogeneous* $SL(2, C)$, for example, in [SW00]. Thus one may call this the *inhomogeneous spin group* $Spin(\mathbb{R}^{1+3})_0$.

The setup on which the analysis in Chapters 3 and 4 was based has to be generalised slightly. To incorporate translations, not only wedges located in the origin (Lorentz transforms of the right wedge), but general wedges have to be considered. So in this section the set of wedges is enhanced to $\mathcal{W}_P := \{PW; P \in \mathcal{P}_+^\uparrow, W \in \mathcal{W}\}$.

Definition 5.1. *The reflection in the edge of the wedge $a \in \mathcal{W}$ was denoted by j_a. If the wedge a is translated by $x \in \mathbb{R}^{1+3}$, then the corresponding wedge is denoted by (a, x), and the reflection in its edge is written $j_{a,x}$.*

The assumption of modular $\mathrm{P}_1\mathrm{CT}$ symmetry then reads

$$J_{a,x} F(\mathfrak{c}, f) J_{a,x} = F^t(C_a \mathfrak{c}, \overline{j_{a,x} f}). \tag{5.3}$$

Note that the translation of the wedge does not affect the involution C_a on the component space. As before, $(j_{a,x} f)(y) = f(j_{a,x} y)$ for $f \in \mathcal{S}$ and $y \in \mathbb{R}^{1+3}$.

5.1 Constructing the Representation

The representation of the universal cover of the Poincaré group will be composed from a representation of the translation subgroup and the representation of the double cover of the Lorentz group constructed in Chapter 4.

Fortunately, a representation of the translation subgroup in terms of modular conjugations has already been achieved by Buchholz, Dreyer, Florig and Summers in [BDFS00, Section 4.3] (cf. also [BS04]). For transferring the representation to the present setting, the following straightforward generalisations of Lemma 2.6 and Lemma 3.6 and their consequences (equations (3.6), (3.7) and (3.8)) are required.

Lemma 5.2. *Let U be a unitary or antiunitary operator in $\mathcal{H}$, (a,x) and (b,y) wedges and recall the definition of $\kappa := \frac{1+ik}{1+i}$.*

(i) If $U\mathbf{F}(a,x)U^\dagger = \mathbf{F}(b,y)$ *then* $UJ_{a,x}U^\dagger = J_{b,y}$.

(ii) $\kappa^\dagger J_{a,x}\kappa = J_{-a,x}$.

(iii) If $U\mathbf{F}(a,x)U^\dagger = \mathbf{F}^t(b,y)$ *then* $UJ_{a,x}U^\dagger = J_{-b,y}$.

(iv) $J_{a,x}J_{b,y}J_{a,x} = J_{j_{a,x}(-b,y)} = J_{(-j_{a,x}b, j_{a,x}y)}$.

The representation of the translation group is defined as follows. Let $a := W_\xi$ be a wedge given by the zweibein ξ and let x be an element of the space spanned by t_ξ and x_ξ. This implies $j_a x = -x$. Then $j_{a,x}j_a = T(2x)$, where $T(2x)$ is the translation by $2x$. In [BDFS00] this was used to define a representation of the translation subgroup of the Poincaré group[1] by setting $\tilde{W}(2x) := J_{a,x}J_a$ for some wedge a with $j_a x = -x$. The definition is independent of the choice of $a \in \mathcal{W}$ obeying the condition mentioned above.

One can combine the representation of the universal cover of the Lorentz group given in the previous chapter with the representation of the translations to get a representation of the universal cover of the proper Poincaré group $\mathcal{P}_+^\uparrow$. In [BS04] a representation of $L_+^\uparrow$ was extended to a representation of the restricted Poincaré group in a similar way. For $x \in T, g \in \tilde{L}_+^\uparrow$ we define

$$\tilde{W}(x,g) := \tilde{W}(x)\tilde{W}(g). \tag{5.4}$$

Then $\tilde{W}$ is a representation because

$$\begin{aligned}\tilde{W}(x,g)\tilde{W}(y,h) &= \tilde{W}(x)\tilde{W}(g)\tilde{W}(y)\tilde{W}(h)\\ &= \tilde{W}(x)\tilde{W}(g)\tilde{W}(y)\tilde{W}(g)^\dagger\tilde{W}(g)\tilde{W}(h)\\ &= \tilde{W}(x)\tilde{W}(\tilde{\lambda}(g)y)\tilde{W}(gh)\\ &= \tilde{W}(x+\tilde{\lambda}(g)y)\tilde{W}(gh)\\ &= \tilde{W}((x,g)(y,h)).\end{aligned}$$

[1] Cf. also [Bor93, Proposition 3.1 and Lemma 3.2] and [Dav95, proof of Theorem 4].

The equality $\tilde{W}(g)\tilde{W}(y)\tilde{W}(g)^\dagger = \tilde{W}(\tilde{\lambda}(g)y)$ follows from the definition of $\tilde{W}$ and Lemma 5.2 (*iv*).

5.2 The Spin-Statistics Theorem and the PCT Operator

We are now able to state the central result of this thesis, the general spin-statistics theorem for quantum fields in four spacetime dimensions under the assumption of modular P_1CT symmetry. The spectrum condition or a unitary representation of the double cover of the Poincaré group given from the outset are not required. Instead, the representation has been constructed in terms of modular conjugations. The number of the components of the quantum field is not restricted to be finite.

Theorem 5.1 (Spin-statistics theorem). *Let F be a quantum field with component space $\mathfrak{C}$ satisfying the generalised Wightman axioms given in Section 2.3.3. Then there exists a canonical unitary and Poincaré covariant representation $\tilde{W}$ together with a representation $\tilde{D}$ of the universal cover of the Lorentz group in the component space $\mathfrak{C}$ which exhibits the spin-statistics connection, i.e.*

$$\tilde{W}(-\mathbb{1}) = k, \qquad \textit{and}$$
$$F_\pm(\mathfrak{c}, f) = \frac{1}{2}\Big(F(\mathfrak{c}, f) \pm F(\tilde{D}(-\mathbb{1})\mathfrak{c}, f)\Big)$$

for all $\mathfrak{c} \in \mathfrak{C}$ and all $f \in \mathcal{S}$. If $\tilde{D}$ is an irreducible representation with spin s, then $\tilde{D}(-\mathbb{1}) = \exp(2\pi i s)$ and one recovers the well-known result

$$F_- = 0 \qquad \textit{for integer spin and}$$
$$F_+ = 0 \qquad \textit{for half-integer spin.}$$

The proof of the spin-statistics theorem for this representation now proceeds exactly as demonstrated in Section 3.3 and in [KL07]. The representation $\tilde{D}$ of $Spin(\mathbb{R}^{1+3})_0$ is given by

$$\tilde{D} : \mathcal{W} \cdot \mathcal{W} \to \mathrm{Aut}(\mathfrak{C}) : \qquad ab \mapsto \tilde{D}(ab) := C_a C_b$$

as before. Theorem 5.1 is achieved in a considerably more general setting than previous ones. In particular, no assumption on the number of components for the field has to be made, the spectrum condition is not required and even the existence of a covariant representation of the universal cover of the Poincaré group is a consequence and not an axiom. Besides normal commutation relations and the usual domain properties for the field operators only modular P_1CT symmetry, which is a consequence of the usual axioms, is required. The covariant representation which is built from quadruples of modular conjugations can naturally be associated with the quantum field.

It remains to elucidate the relation of modular P_1CT symmetry to the existence of a full PCT operator. As already remarked in [Kuc05], in four spacetime dimensions a PCT operator Θ can be defined as the product of three modular conjugations. Choose an orthonormal basis e_i; $i = 0, \ldots, 3$, of $\mathbb{R}^{1+3}$ with e_0 timelike. Then the three wedges $a_i := W_{(e_0,e_i)}, i = 1, 2, 3$, have associated modular conjugations J_{a_i}. The operator $\Theta := J_{a_1} J_{a_2} J_{a_3}$ implements a reflection in parity, time and charge. It has the following properties.

Theorem 5.2. *The operator Θ has the following properties*

(i) Θ *is antiunitary.*

(ii) $\Theta \tilde{W}(x, g)\Theta^{-1} = \tilde{W}(-x, g)$.

(iii) Θ^2 *commutes with bosonic components and anticommutes with fermionic components of the field.*

(iv) Θ *depends only on the orientation of the orthonormal set* e_1, e_2, e_3. *The PCT operator for another orientation differs from* Θ *by* k.

Proof. The first property follows by inspection. To prove the second choose $a, b, c \in \mathcal{W}$ with $\tilde{W}(g) = J_a J_b$ and $\tilde{W}(x) = J_{(c,x)} J_c$. Then $J_{a_1} J_{a_2} J_{a_3} \tilde{W}(x, g) J_{a_3} J_{a_2} J_{a_1} = J_{(-c,-x)} J_{-c} J_{-a} J_{-b} = \tilde{W}(-x) J_a J_b k^2 = \tilde{W}(-x, g)$ because $j_{a_1} j_{a_2} j_{a_3} = j_{e_0} j_{e_1} j_{e_2} j_{e_3} = -1 \in L_+$. For the third property check $a_1 a_2 a_3 a_1 a_2 a_3 = -\mathbb{1}$ by using $a_i a_j = -a_j a_i$ for $i \neq j$. This implies $\Theta^2 = J_{a_1} J_{a_2} J_{a_3} J_{a_1} J_{a_2} J_{a_3} = \tilde{W}(a_1 a_2 a_3 a_1 a_2 a_3) = \tilde{W}(-\mathbb{1}) = k$, and the statement follows by definition of $F_\pm$ and $k^2 = \mathbb{1}$. Let the orthonormal vectors $e'_1, e'_2, e'_3 \in e_0^\perp$ have the same orientation as e_1, e_2, e_3 and set $a'_i := W_{(e_0,e'_i)}$ and $\Theta' := J_{a'_1} J_{a'_2} J_{a'_3}$. Then there exists a continuous path $R(t)_{t\in[0,1]}$ in the rotation group with $R(0) = 1$ and $R(1)e_i = e'_i$ for $i = 1, 2, 3$. Now define a map $[0, 1] \ni t \mapsto \gamma(t) := a_3 a_2 a_1 (R(t)a_1)(R(t)a_2)(R(t)a_3)$. Because of $\widetilde{Ad} \circ \gamma(t) = 1$ for $t \in [0, 1]$ one has $\gamma(t) = \pm\mathbb{1}$. $\gamma(0) = \mathbb{1}$ and continuity of γ imply $\gamma(t) = \mathbb{1}$ for $t \in [0, 1]$. Finally the statement follows because of $\Theta^{-1}\Theta' = J_{a_3} J_{a_2} J_{a_1} J_{a'_1} J_{a'_2} J_{a'_3} = \tilde{W}(\gamma(1)) = \tilde{W}(\mathbb{1}) = \mathbb{1}$. The change of orientation introduces a factor k, since $J_{-e_1} J_{-e_2} J_{-e_3} = k J_{e_1} J_{e_2} J_{e_3}$. □

5.3 Transfer of the Result to Algebraic Quantum Field Theory

It is straightforward to transfer the results to case of a field algebra associated with a net of observables in AQFT as in [GL95, Section 2]. So let $\mathcal{O} \to \mathbf{F}(\mathcal{O})$ be a net of algebras of bounded operators. As before, the statistics operator is denoted by

k and $\kappa = \frac{1+ik}{1+i}$. Assume normal commutation relations to hold. If modular $\mathrm{P}_1\mathrm{CT}$ symmetry holds in the form

$$J_a \mathbf{F}(\mathcal{O}) J_a = \mathbf{F}^t(j_a \mathcal{O}),$$

then this net obeys the generalised Wightman axioms of Section 2.3.3 if one ignores the component space. The construction of the representation of the universal cover of the Poincaré group proceeds exactly as presented in Chapter 4 and above. The property

$$\tilde{W}(-\mathbb{1}) = J_a J_{-a} = k$$

for every $a \in \mathcal{W}$ remains unaffected, and this is the spin-statistics theorem for this setting.

As mentioned before, Guido and Longo [GL95] also proved an algebraic spin-statistics theorem for the field algebras. They started from the assumption of modular covariance and constructed a representation of the universal cover of the Poincaré group in terms of the modular unitaries Δ_a^{it} for $a \in \mathcal{W}$ and $t \in \mathbb{R}$. So it is natural to compare these representations. The following lemma [KL07] shows that the two representations in fact coincide.

Lemma 5.3. *Assume modular P_1CT symmetry and assume that the modular unitaries generate a representation acting covariantly on the net of field algebras. Then the representation $\tilde{W}$ and the representation U defined in [GL95] in terms of modular unitaries coincide.*

Proof. It suffices to show that for each Δ_a^{it}, $a \in \mathcal{W}$, $t \in \mathbb{R}$, there exist $b, c \in \mathcal{W}$ such that $\Delta_a^{it} = J_b J_c$.

Choose a zweibein (e_0, e_1) with $a = e_0 \cdot e_1$ and e_0 future directed. Then there exists a zweibein (e_0, e_2) with e_2 orthogonal to e_1. Set $b := e_0 \cdot e_2$. Now it follows from $j_a b = b$ and Lemma 3.6 that $J_b \Delta_a^{it} J_b = \Delta_a^{-it}$ and by modular covariance $\Delta_a^{-it/2} J_b \Delta_a^{it/2} = J_{\Lambda_a(t/2)b}$ for all $t \in \mathbb{R}$. $(\Lambda_a(t))_{t\in\mathbb{R}}$ is the one-parameter group of boosts leaving the wedge a invariant (cf. eq. (2.2)). This entails

$$J_b \Delta_a^{it} = \Delta_a^{-it/2} J_b \Delta^{it/2} = J_{\Lambda_a(-t/2)b}, \quad \text{i.e.} \quad \Delta_a^{it} = J_b J_{\Lambda_a(-t/2)b}. \qquad \square$$

The results which have been established before ([GL95] and references therein) under the assumption of modular covariance guarantee the existence of a representation. The advantage of the representation $\tilde{W}$ defined by quadruples of modular conjugations is that the unitary operators $\tilde{W}(g)$ for $(x, g) \in \mathcal{P}_+^\uparrow$ can be given explicitly. What remains an open question is the case of spacetime dimensions different from four.

6 Conclusion

The Bisognano-Wichmann theorem states that in the usual Wightman framework of quantum field theory the modular conjugations and modular groups associated with wedge regions have a geometrical interpretation. These geometrical actions are referred to as modular $\mathrm{P}_1\mathrm{CT}$ symmetry and modular covariance, respectively.

Based on modular $\mathrm{P}_1\mathrm{CT}$ symmetry a spin-statistics theorem for a quantum field with rotation symmetry has been derived in [Kuc05] from very general physical principles. No covariant representation of the symmetry group is assumed from the outset, the spectrum condition is not required, and the Wightman quantum field is not restricted to have only finitely many components. The essential assumptions are cyclicity of the vacuum, normal commutation relations and modular $\mathrm{P}_1\mathrm{CT}$ symmetry. A covariant unitary representation of the universal cover of the rotation group, for which the spin-statistics theorem holds, is constructed from pairs of modular conjugations. In addition, a PCT operator can be defined.

This analysis was presented in Chapter 3 in an improved way, using properties of the spin group $Spin(\mathbb{R}^3)_0$, which is the universal covering of $SO(3)$. The extension of this result to Lorentz- or Poincaré-invariant theories is the main result achieved in this thesis and is presented in Chapters 4 and 5. A major task is a suitable characterisation of elements in the universal covering of the Lorentz group by pairs of wedges. This is needed to assign to an element of the covering group the product of two modular conjugations associated with wedges in an adequate way to obtain a representation.

Two solutions to this problem are given, which are closely related in spite of their different starting points. One approach starts from the set of pairs of wedges and yields a set $\mathbf{G}_L$ by introducing an equivalence relation and considering the quotient space. This set is shown to fulfil the necessary requirements of the universal covering of the Lorentz group with a natural covering map. Although the definition of $\mathbf{G}_L$ is intuitive, establishing that the covering map is indeed open and a local homeomorphism is involved. The second and more elegant solution is based on the well-known identification of $Spin(\mathbb{R}^{1+3})_0$ with a subgroup of the Clifford algebra associated with an orthogonal space. Parts of the results achieved in the first approach enter here again in the description of $Spin(\mathbb{R}^{1+3})_0$ by pairs of wedges.

Either of these realisations of the universal covering of the Lorentz group can be used to define a unitary representation thereof in terms of pairs of modular conjugations. For the bosonic case a similar problem has been studied by Buchholz and Summers [BS04], and their proofs have been generalised to the setting under consideration. With the representation on the Hilbert space at hand it is straightforward to define a representation on the component space of the field. The proof of the spin-statistics theorem then proceeds as in [Kuc05], and also the definition of the PCT operator can be easily transferred. For a field transforming under an irreducible representation with spin s one recovers the usual spin-statistics theorem which forbids fermionic fields to obey commutation relations and bosonic fields to obey anticommutation relations.

The approach taken here is in the spirit of Belinfante [Bel39], who assumed invariance under charge conjugation, and Schwinger [Sch51], who derived the spin-statistics theorem from the PCT theorem. It should be mentioned that the proof does not depend on analyticity properties of the vacuum expectation values and is in some sense of a more algebraic nature.

For a net of algebras of observables in algebraic quantum field theory, there exists a unique field algebra obeying normal commutation relations and a unique gauge group [DR90]. The assumptions stated in Section 2.3.3 are general enough to adapt the analysis in a straightforward way to the case of a net of field algebras. This allows for a comparison of the results presented here with similar work of Guido and Longo. These authors derived an algebraic spin-statistics theorem as well as a PCT operator from the assumption of modular covariance. If we additionally assume this property, then the representations originating from modular P_1CT symmetry and modular covariance coincide. The counterexamples of Streater [Str67] to the spin-statistics theorem and Oksak and Todorov [OT68] to the PCT theorem for fields with infinitely many components are based on specific choices of representations of the universal covering group of $L_+^\uparrow$. Constructing the representation from modular objects provides a natural and canonical method to select one with desirable features.

The arguments used to prove the representation to be well-defined crucially depend on the spacetime dimension. One example is the fact that in four spacetime dimensions the group of Lorentz transformations leaving the edge of a wedge invariant is abelian. This does not hold for higher spacetime dimensions. Nevertheless, the analysis of such cases or other spacetimes would be interesting and may yield further insights. Another interesting problem is the relation of the two assumptions of modular P_1CT symmetry and modular covariance. Guido and Longo [GL95] showed that modular covariance implies modular P_1CT symmetry. In [Dav95] the question was studied under an additional assumption of additivity of the net and a stronger cyclicity assumption than in our setting, and equivalence was concluded. But this does not hold in general, as there are examples for local nets of algebras of observables satisfying modular P_1CT symmetry but not modular covariance [BDFS00, Chapter 5.3]. In these counterexamples the unitary representation acting on the net may even satisfy the spectrum condition. In the representation obtained from the assumption of modular covariance [GL95] only the unitary operators implementing boosts are explicitly defined. An advantage of the approach taken here is that the complete representation is explicitly given.

A C*-Algebras, States and Representations

Some basic notions of the theory of C*-algebras will be introduced here to keep the text self-contained. The material can be found in standard textbooks like [BR79, KR97a, Tak02]

A *-algebra is an algebra equipped with an involutive antilinear $*$-operation satisfying $(AB)^* = B^*A^*$ for all elements A, B. An algebra is normed if the norm satisfies besides the usual properties of a norm on a vector space $||AB|| \leq ||A||\,||B||$.

A C*-algebra $\mathcal{A}$ is a normed complete *-algebra (usually over $\mathbb{C}$) whose norm satisfies the C*-property. This means that for $A, B \in \mathcal{A}$ the equation

$$||A^*A|| = ||A||^2$$

holds. An example of a C*-algebra is the algebra $\mathcal{B}(\mathcal{H})$ of bounded operators on a Hilbert space with the usual operator norm and the adjoint as $*$-operation. A basic structure theorem states that in fact every C*-algebra is isomorphic to a norm-closed selfadjoint algebra of bounded operators on some $\mathcal{H}$.

In the following it will be assumed that $\mathcal{A}$ has a unit $\mathbb{1}$. A state ω on $\mathcal{A}$ is a positive linear functional over $\mathcal{A}$ with $\omega(\mathbb{1}) = 1$. A functional ω is positive if $\omega(A^*A) \geq 0$ for all $A \in \mathcal{A}$. And a state ω is faithful if $\omega(A^*A) > 0$ for $A \neq 0$.

If $(\mathcal{H}, \pi)$ is a nondegenerate representation of the C*-algebra $\mathcal{A}$ and $\Omega \in \mathcal{H}$ with $||\Omega|| = 1$, then Ω defines a state on $\mathcal{A}$ by

$$\omega_\Omega(A) := \langle \Omega, \pi(A)\Omega \rangle. \tag{A.1}$$

Conversely, for every state ω on a C*-algebra $\mathcal{A}$, one can define a representation π_ω of $\mathcal{A}$ in a Hilbert space $\mathcal{H}_\omega$ so that ω is defined via a unit vector Ω_ω as in equation (A.1). The construction of the triple $(\pi_\omega, \mathcal{H}_\omega, \Omega_\omega)$ is called GNS-construction.

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