Remarks on the expected stress-energy tensor of quantized Dirac fields on curved backgrounds

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Motivation

Quantum fields on curved spacetimes

- Quantum Field Theory on Curved Spacetimes (QFT on CST): approximate solution to the problem of formulating a quantum theory of both gravity and matter
- Matter: quantum fields
- Spacetime: arbitrary but fixed classical curved background, non-dynamical in particular

Curved spacetimes from quantum fields

- Back-reaction of the quantum field on the (curvature of) spacetime

$$G_{\mu\nu}(x) = 8\pi G \omega(: T_{\mu\nu}(x) :)$$

- This can be formally derived by expanding around a vacuum solution, keeping "one-loop" ($\hbar^1$) terms of the quantum matter and "tree" ($\hbar^0$) terms of the quantum metric ...

- ... and can thus only make sense for special states or as a model equation.

- It also seems necessary to quantise matter "on all spacetimes at once".

In this talk

- How can one sensibly define a r.h.s. for $G_{\mu\nu}(x) = 8\pi G\omega(: T_{\mu\nu}(x) :)$?
- We will see that in the case of Dirac spinor fields

  1. a modified version of the classical stress-energy tensor,
  2. regularised by point-splitting and subtraction of the Hadamard singularity
  3. and evaluated on Hadamard states gives a satisfactory result.

Outline of the talk

1. Classical free Dirac fields on curved spacetimes
2. Quantisation in the framework of AQFT
3. Hadamard states
4. The expected stress-energy tensor
5. Conclusions

Classical free Dirac fields on curved spacetimes

The Dirac field on Minkowski spacetime

- Global Poincaré invariance
- Dirac spinor field

$$\psi : (\mathbb{R}^4, \eta) \longrightarrow \mathbb{C}^4,$$

transforming covariantly under the $D^{(\frac{1}{2},0)} \oplus D^{(0,\frac{1}{2})}$ representation of

$$SL(2,\mathbb{C}) \simeq Spin^0(3,1) \twoheadrightarrow \mathcal{L}_+^{\uparrow} = SO^0(3,1)$$

- Dirac equation: determined by transformation properties and irreducibility of the representation

The Dirac field on curved spacetimes

- Spacetime: (M, g) is a fourdimensional, Hausdorff, globally hyperbolic, smooth manifold M with smooth Lorentzian metric g of signature $(-,+,+,+)$.
- We have only local Lorentz invariance, we thus can only
  1. describe the Dirac field ψ as a section of a $\mathbb{C}^4$-bundle ($\longrightarrow$ Dirac bundles),
  2. assure a globally consistent local double-covering $Spin^0(3,1) \twoheadrightarrow \mathcal{L}_+^\uparrow$ to define sensible transformation properties of ψ ($\longrightarrow$ spin structure)
  3. and take the generally covariant generalisation of the Minkowskian Dirac equation ($\longrightarrow$ spin connection, γ-matrices).

The γ-matrices

- The matrices $\{\gamma_a\}_{a=0..3} \subset M(4,\mathbb{C})$ constitute a complex irreducible representation π_{Cl} of $Cl(3,1)$, i.e.

$$\{\gamma_a, \gamma_b\} \doteq \gamma_a\gamma_b + \gamma_b\gamma_a = 2\eta_{ab}\mathbf{1}.$$

- In the following: choose an arbitrary but fixed π_{Cl}

- Unique Dirac conjugation matrix $\beta \in SL(4,\mathbb{C})$:

$$\beta^* = \beta \qquad \gamma_a^* = -\beta\gamma_a\beta^{-1}$$

$$i\beta n^a\gamma_a > 0 \text{ for } n \text{ future pointing \& timelike}$$

The spin group

- $Spin^0(3,1) \subset Cl(3,1)$ and can thus be (reducibly) represented on $\mathbb{C}^4$ via π_{Cl}.
- The double covering $\Lambda : Spin^0(3,1) \twoheadrightarrow \mathcal{L}_+^\uparrow$ can be specified as

$$S\gamma_a S^{-1} = \gamma_b \Lambda^b{}_a(S).$$

Spin structures

- In the following: Let (M, g) be oriented and time-oriented.
- The Lorentz frame bundle $LM(\mathcal{L}_+^\uparrow, \pi_L, R_L, M)$ is the collection of all orthogonal frames $\{e_a\}_{a=0..3}$, $g(e_a, e_b) = \eta_{ab}$ of M.
- A spin structure on (M, g) is a pair (SM, ρ), where $SM(Spin^0(3,1), \pi_S, R_S, M)$ is the bundle of spin frames and $\rho : SM \to LM$ s.t.

$$\rho \circ \pi_S = \pi_L \circ \rho \quad \text{and} \quad \rho \circ R_S(S) = R_L(\Lambda(S)) \circ \rho.$$

- On globally hyperbolic, fourdimensional (M, g) spin structures always exist. [Geroch]

Dirac fields I

- Dirac bundle: $DM \doteq SM \times_{\pi_{Cl}} \mathbb{C}^4$
- Dual Dirac bundle: D^*M, dual w.r.t. (an extension of) the Hermitian inner product on $\mathbb{C}^4$
- Spaces of smooth sections (with compact support): $\mathcal{E}(M, DM)$, $\mathcal{E}(M, D^*M)$, $\mathcal{D}(M, DM)$, $\mathcal{D}(M, D^*M)$
- Classical Dirac spinor field: $\psi \in \mathcal{E}(M, DM)$ (column vector)
- Classical Dirac cospinor field: $\psi' \in \mathcal{E}(M, D^*M)$ (row vector)

Dirac fields II

- Global pairing of $\mathcal{D}(M, DM)$ and $\mathcal{E}(M, D^*M)$ or $\mathcal{E}(M, DM)$ and $\mathcal{D}(M, D^*M)$

$$\langle \psi' \psi \rangle \doteq \int_M d\mu_g(x)\, \psi'(x)\, (\psi(x))$$

- Dirac conjugation

$$^\dagger : \mathcal{E}(M, DM) \to \mathcal{E}(M, D^*M), \quad \psi^\dagger(x) \doteq \psi(x)^* \beta$$

$$^\dagger : \mathcal{E}(M, D^*M) \to \mathcal{E}(M, DM), \quad \psi'^\dagger(x) \doteq \beta^{-1} \psi'(x)^*$$

Spinor-tensors

- Mixed tensor products of TM, T^*M, DM and D^*M yield trivial bundles!
- Sections of these bundles can be expressed via global frames

$$E \doteq \{E_A\}_{A=1..4}, \quad \{E^B\}_{B=1..4}, \quad E^B(E_A) = \delta^B_A,$$
$$e \doteq \{e_a\}_{a=0..3}, \quad \{e^b\}_{b=0..3}, \quad g(e^b, e_a) = \delta^b_a, \quad e = \rho \circ E.$$

- Example: $\mathcal{E}(M, T^*M \otimes DM \otimes D^*M) \ni \gamma \doteq \gamma^A_{aB}\ e^a \otimes E_A \otimes E^B$
- Switching from $e_a = e^\mu_a \partial_\mu,\ e^b = e^b_\nu d^\nu$ to $\partial_\mu,\ d^\nu$ via $e^\mu_a,\ e^b_\nu$

The spin connection

- Connection on SM: pullback of Levi-Civita connection on LM via ρ
- Covariant derivative on spinor-tensors, e.g.:

$$\nabla_{e_a}\gamma^{A}_{bB} \doteq \nabla_a\gamma^{A}_{bB} \doteq \gamma^{A}_{bB;a} = 0$$

- Spin curvature tensor C_{ab}

The Dirac equations

- Feynman slash notation $\not{v} \doteq v^a \gamma_a$
- Dirac operators

$$D^{(\prime)} : \mathcal{E}(M, DM) \to \mathcal{E}(M, DM), \quad D^{(\prime)} : \mathcal{E}(M, D^*M) \to \mathcal{E}(M, D^*M),$$
$$D \doteq -\not{\nabla} + m, \quad D' \doteq \not{\nabla} + m$$

- Dirac equations for $\psi \in \mathcal{E}(M, DM)$, $\psi' \in \mathcal{E}(M, D^*M)$

$$D\psi = 0, \quad D'\psi' = 0 \quad (1)$$

- Solutions of (1) solve the spinorial Klein-Gordon equation [Lichnerowicz]

$$P\psi^{(\prime)} = 0, \quad P \doteq -D'D = -DD' = \nabla_a \nabla^a - \frac{R}{4} - m^2.$$

Quantisation in the framework of AQFT

The fundamental solutions of D and D'

- Unique fundamental solutions of D and D' [Dimock]

1. $$S^{\pm} : \mathcal{D}(M, DM) \to \mathcal{E}(M, DM), \quad S^{\pm}_{*} : \mathcal{D}(M, D^{*}M) \to \mathcal{E}(M, D^{*}M)$$
   $$DS^{\pm} = S^{\pm}D = id_{\mathcal{D}(M,DM)}, \quad D'S^{\pm}_{*} = S^{\pm}_{*}D' = id_{\mathcal{D}(M,D^{*}M)}$$
2. $$\text{supp}\,(S^{\pm}u) \subset J^{\pm}(\text{supp}\ u)\ \forall u \in \mathcal{D}(M, DM),$$
   $$\text{supp}(S^{\pm}_{*}v) \subset J^{\pm}(\text{supp}\ v)\ \forall v \in \mathcal{D}(M, D^{*}M)$$
3. $S^{\pm} = D'E^{\pm}, \quad S^{\pm}_{*} = DE^{\pm}_{*}$, where $E^{\pm}$ and $E^{\pm}_{*}$ are the fundamental solutions of $-P$.
4. Causal propagators

   $$S \doteq S^{+} - S^{-}, \quad S_{*} \doteq S^{+}_{*} - S^{-}_{*}$$

Doubling the fields

- Doubling the fields turns out to be convenient for defining the field algebra. [Köhler, Fewster & Verch, ...]
- $\widetilde{\mathcal{D}}(M) \doteq \mathcal{D}(M, DM) \oplus \mathcal{D}(M, D^*M), \quad \widetilde{S} \doteq S \oplus S_*, \quad \widetilde{D} \doteq D \oplus D'$
- Conjugation $\Gamma : \widetilde{\mathcal{D}}(M) \to \widetilde{\mathcal{D}}(M), \quad u \oplus v \mapsto v^\dagger \oplus u^\dagger$
- Positive definite sesquilinear product on $\widetilde{\mathcal{D}}(M)/\ker \widetilde{S} \ni [f_i] \doteq [u_i \oplus v_i]$

$$([f_1],[f_2]) \doteq -i \left\langle u_1^\dagger \, S(u_2) \right\rangle + i \left\langle S_*(v_1) \, v_2^\dagger \right\rangle$$

- Hilbert space $\mathcal{H} \doteq \overline{\widetilde{\mathcal{D}}(M)/\ker \widetilde{S}}$

The field algebra

- $\mathcal{F}(M) \doteq \mathrm{CAR}(\mathcal{H}, \Gamma)$ [Araki, Fewster & Verch]: Unique C^*-algebra generated by $\mathbf{1}$ and $\{B(f) : f \in \mathcal{H}\}$ subject to
  1. $f \mapsto B(f)$ is $\mathbb{C}$-linear
  2. $B(\Gamma f) = B(f)^*$
  3. $\{B(f_1)^*, B(f_2)\} = (f_1, f_2)\mathbf{1}$ (CAR)
  4. implicit: $B(\widetilde{D} f) = 0$ (EOM)
- Equivalently: $\mathcal{F}(M) = \overline{\mathcal{F}_0(M)}$ (Borchers-Uhlmann algebra) [Köhler, Sanders]

$$\mathcal{F}_0(M) \doteq \bigoplus_{n=0}^{\infty} \mathcal{D}\left((DM \oplus D^* M)^{\boxtimes n} \right) / (\text{ideal generated by EOM \& CAR})$$

Back to single fields

- Dirac (co)spinor quantum fields

$$\psi(v) \doteq B(0 \oplus v), \quad \psi^\dagger(u) \doteq B(u \oplus 0)$$

- $\psi(v)^* = \psi^\dagger(v^\dagger)$
- $\{\psi(v), \psi^\dagger(u)\} = -i \langle v\, S(u)\rangle \mathbf{1}$ and all other anticommutators vanish.
- $D\psi(v) \doteq \psi(D'v) = 0, \quad D'\psi^\dagger(u) \doteq \psi^\dagger(Du) = 0$

The algebra of observables

- Let supp u and supp v as well as supp $u_1 \cup$ supp v_1 and supp $u_2 \cup$ supp v_2 be spacelike separated:

$$\{\psi(v), \psi^\dagger(u)\} = -i \langle v\, S(u) \rangle \mathbf{1} = 0 \quad \text{but e.g.}$$

$$[\psi^\dagger(u_1)\psi(v_1), \psi^\dagger(u_2)\psi(v_2)] = \cdots = 0.$$

- Possible algebra of observables

$$\mathcal{A}(M) \doteq \text{even subalgebra of } \mathcal{F}(M)$$

- But $\mathcal{A}(M)$ is both "too large" and "too small", one needs to include Wick polynomials and restrict to "gauge-invariant" elements.

Locality and general covariance

- Locally covariant QFT [..., Dimock, Kay, Hollands & Wald, Verch, Brunetti & Fredenhagen & Verch, Fewster, Sanders, ...]
- The Dirac field B $(\psi, \psi^\dagger)$ is locally covariant [Sanders]. Essentially, let

$$\chi : (M_1, g_1, SM_1, \rho_1) \mapsto (M_2, g_2, SM_2, \rho_2)$$

be a map which

1. corresponds to an isometric embedding of (M_1, g_1) into (M_2, g_2),
2. preserves space and time orientation as well as causal relations
3. and respects the spin structure,

then $\exists$ an injective, unit-preserving *-homomorphism $\alpha_\chi : \mathcal{F}(M_1) \to \mathcal{F}(M_2)$ s.t. B can be understood as a collection of continuous maps

$$B_M : \widetilde{\mathcal{D}}(M) \mapsto \mathcal{F}(M), \quad \alpha_\chi \circ B_{M_1} = B_{M_2} \circ \chi_* .$$

Hadamard states

Quasifree states

- Quasifree, gauge-invariant state ω on $\mathcal{A}(M)$

  1. $\omega : \mathcal{A}(M) \to \mathbb{C}$ linear
  2. $\omega(A^* A) \geq 0 \quad \forall A \in \mathcal{A}(M), \quad \omega(\mathbf{1}) = 1$
  3. $$\omega\left(\psi^\dagger(u_1)\cdots\psi^\dagger(u_m)\psi(v_1)\cdots\psi(v_n)\right) = \delta_{mn} \sum_{\pi_m \in S_m} \prod_{i=1..m} \text{sign}(\pi_m)\omega\left(\psi^\dagger(u_i)\psi(v_{\pi_m(i)})\right)$$

- $\omega^+(u,v) \doteq \omega\left(\psi(v)\psi^\dagger(u)\right) \quad \omega^-(u,v) \doteq \omega\left(\psi^\dagger(u)\psi(v)\right)$

Preferred states

- Minkowski: isometry group (Poincaré group) & spectrum condition $\Rightarrow$ unique vacuum state
- CST: trivial isometry group & microlocal spectrum condition (μSC) $\Rightarrow$ Hadamard states
- Properties of Hadamard states:
  1. same UV behaviour as the Minkowski vacuum [Radzikowski, Köhler, Kratzert, Hollands, Sahlmann & Verch]
  2. physically equivalent, i.e. quasiequivalent [Verch, Hollands]
  3. ...
  4. well-suited for a definition of $\omega(:T_{\mu\nu}:)$ [Wald]

Bitensors

- Bitensor: (distributional) section of an exterior tensor product bundle, e.g $\omega^{\pm} \in \mathcal{D}'(D^*M \boxtimes DM)$
- Primed index notation: $\omega^{\pm}(x,y) = \omega^{\pm}_{A}{}^{B'}(x,y)\, E^{A}(x) \otimes E_{B'}(y)$.
- Synge's bracket notation: $[F(x,y)_{A}{}^{B'}] \doteq F(x,x)_{A}{}^{B}, \quad F \in \mathcal{E}(D^*M \boxtimes DM)$
- Smooth bitensors on a geodesically convex normal neighbourhood
  1. $\frac{1}{2}$ squared geodesic distance $\sigma(x,y)$
  2. vector parallel transport $g(x,y)^{\mu}_{\nu'}$
  3. spinor parallel transport $I(x,y)^{A}{}_{B'}$

The Hadamard form

- $\omega^{\pm}(x,y)$ are of the Hadamard form $\Leftrightarrow$ $\exists$ smooth bitensors U, V and W, s.t.

$$\omega^{\pm}(x,y) = \pm\frac{1}{8\pi^2} D'_y \left(H^{\pm}(x,y) + W(x,y)\right),$$

$$H^{\pm} \doteq \frac{U}{\sigma} + V \ln\left(\frac{\sigma}{\lambda^2}\right),$$

$$V = \sum_n V_n \sigma^n, \quad W = \sum_n W_n \sigma^n$$

- The definition needs to be refined to avoid convergence problems of V and W, obtain a well-defined distribution and rule out spacelike singularities. [Kay & Wald, Köhler, Verch]

The microlocal spectrum condition

- $\omega^{\pm}(x,y)$ fulfil the μSC $\Leftrightarrow$

$$WF(\omega^{\pm}) = \left\{(x,k_x,y,-k_y) \in (T^*M)^{\boxtimes 2} \setminus \{\mathbf{0}\}, \mid (x,k_x) \sim (y,k_y),\ k_x \underset{\triangleright}{\overset{\triangleleft}{}} 0\right\}$$

- $\omega^{\pm}(x,y)$ fulfil the μSC iff they are of Hadamard form. [Kratzert, Hollands, Sahlmann & Verch]

Determining the Hadamard coefficients

- $D'_x \omega^\pm = D_y \omega^\pm = 0 \quad \Rightarrow \quad D'_x D'_y H$, $P_y H$ smooth (H denotes either H^+ or H^-)

- We have furthermore been able to prove that

Proposition

$(D'_x - D_y)H$ and $P_x H$ are smooth (but non-vanishing).

- This yields recursive differential equations for U, V and W. Starting with $[U] = \mathbf{1}$, one can show:

  1. $U = uI^*$, where u is the Hadamard coefficient of the scalar Hadamard form.
  2. V is not proportional to the scalar Hadamard coefficient v.
  3. U and V depend only on the local curvature and m, while W depends on the state ω.

Coincidence point limits of H

Proposition

The Hadamard bidistributions H fulfil

$$[V_1] = \left(\frac{m^4}{8} + \frac{m^2 R}{48} + \frac{R^2}{1152} - \frac{\Box R}{480} - \frac{R_{\mu\nu}R^{\mu\nu}}{720} + \frac{R_{\mu\nu\rho\tau}R^{\mu\nu\rho\tau}}{720}\right)\mathbf{1} + \frac{C_{\mu\nu}C^{\mu\nu}}{48},$$

$$[P_x H] = 6[V_1], \qquad [(P_x H)_{;\mu}] = 8[V_{1;\mu}], \qquad [(P_x H)_{;\mu'}] = -8[V_{1;\mu}] + 6[V_1]_{;\mu},$$

$$[P_y H] = 6[V_1], \qquad [(P_y H)_{;\mu}] = 8[V_{1;\mu}] - 2[V_1]_{;\mu}, \qquad [(P_y H)_{;\mu'}] = -8[V_{1;\mu}] + 8[V_1]_{;\mu},$$

$$Tr[D'_x D'_y H] = -Tr[P_x H], \qquad Tr[(D'_x D'_y H)_{;\mu}] = -Tr[(P_x H)_{;\mu}] + [V_1]_{;\mu},$$

$$Tr[(D'_x D'_y H)_{;\mu'}] = -Tr[(P_x H)_{;\mu'}] - [V_1]_{;\mu},$$

$$Tr[(P_y H - P_x H)_{;\nu'}]\gamma^\nu \gamma_\mu = 2Tr[V_1]_{;\mu}.$$

- Proof: seven months of calculations, to give you a flavor note that

$$[\sigma_{\alpha\beta\gamma\delta\varepsilon\phi\lambda}] = -\frac{1}{6}R_{\alpha\beta\gamma\delta;\varepsilon\phi\lambda} + 779 \text{ terms.}$$

The expected stress-energy tensor

The classical stress-energy tensor

- Action functional of Dirac fields

$$S[\psi] = \int_{M^4} d^4x \sqrt{|g|} L(\psi) = \int_{M^4} d^4x \sqrt{|g|} \left[\frac{1}{2} \psi^\dagger \left(D\psi \right) + \frac{1}{2} \left(D' \psi^\dagger \right) \psi \right]$$

- Cassical stress-energy tensor of Dirac fields

$$T_{\mu\nu} \doteq \frac{1}{\sqrt{|g|}} \frac{\delta S}{\delta g_{\mu\nu}} = \frac{1}{2} \left(\psi^\dagger \gamma_{(\mu} \psi_{;\nu)} - \psi^\dagger_{;(\mu} \gamma_{\nu)} \psi \right) - \frac{1}{2} L(\psi) g_{\mu\nu}$$

- Dirac equations $\Rightarrow$

$$\nabla^\mu T_{\mu\nu} = 0 \qquad g^{\mu\nu} T_{\mu\nu} = -m \psi^\dagger \psi$$

Why are Hadamard states good for $\omega(: T_{\mu\nu} :)$?

- $G_{\mu\nu}(x) = 8\pi G\omega(: T_{\mu\nu}(x) :)$ can only make sense if $\omega(: T_{\mu\nu}(x) :)$ is finite and $: T_{\mu\nu}(x) :$ has finite fluctuations!
- Possible definition of $\omega(: T_{\mu\nu}(x) :)$: Augment $\mathcal{A}(M)$ with Wick products normal ordered w.r.t. $D'_y H$ and indentify $: T_{\mu\nu}(x) :$ as an element of this enlarged algebra.
- This yields smooth expectation values and finite fluctuations of $: T_{\mu\nu}(x) :$ on all Hadamard states.

Definition of $\omega(:T_{\mu\nu}(x):)$

- Here we are only interested in $\omega(:T_{\mu\nu}(x):)$: employ a straightforward definition without enlarging $\mathcal{A}(M)$
- Point-splitting along a geodesic

$$T_{\mu\nu}(x,y) \doteq \frac{1}{2}\left(\psi^\dagger(x)\gamma_{(\mu}g^{\nu'}_{\nu)}\psi(y)_{;\nu'} - \psi^\dagger(x)_{;(\mu}\gamma_{\nu)}\psi(y)\right)$$

- Subtraction of the singularity, conininciding point limit

$$\begin{aligned}\omega(:T_{\mu\nu}(x):) &\doteq Tr\left[\omega(T_{\mu\nu}(x,y)) - T^{sing}_{\mu\nu}(x,y)\right] \\ &\doteq Tr\left[D^0_{\mu\nu}\left(\omega^-(x,y) + \frac{1}{8\pi^2}D'_y H\right)\right] \doteq \frac{1}{8\pi^2}Tr\left[D_{\mu\nu}W(x,y)\right]\end{aligned}$$

- Canonical but unsatisfactory choice of $D^0_{\mu\nu}$, $D_{\mu\nu}$

$$D^{0,can}_{\mu\nu} \doteq \frac{1}{2}\gamma_{(\mu}\left(g^{\nu'}_{\nu)}\nabla_{\nu'} - \nabla_{\nu)}\right) \qquad D^{can}_{\mu\nu} \doteq -D^{0,can}_{\mu\nu}D'_y$$

Wald's axioms I

- (A1) Given ω_1 and ω_2, such that $\omega_1^-(x,y) - \omega_2^-(x,y)$ is smooth,

$$\omega_1(:T_{\mu\nu}(x):) - \omega_2(:T_{\mu\nu}(x):) = Tr\left[D^{0,can}_{\mu\nu}\left(\omega_1^- - \omega_2^-\right)\right].$$

- (A2) $\omega(:T_{\mu\nu}(x):)$ is locally covariant: Let

$$\chi : (M_1, g_1, SM_1, \rho_1) \mapsto (M_2, g_2, SM_2, \rho_1),$$

$$\alpha_\chi : \mathcal{A}(M_1) \to \mathcal{A}(M_2)$$

as before. If two states ω_1 and ω_2 on $\mathcal{A}(M_1)$ and $\mathcal{A}(M_2)$ are related by $\omega_1 = \omega_2 \circ \alpha_\chi$, then

$$\omega_2(:T_{\mu_2\nu_2}(x_2):) = \chi_*\left(\omega_1(:T_{\mu_1\nu_1}(x_1):)\right).$$

Wald's axioms II

- (A3) $\nabla^\mu \omega(: T_{\mu\nu}(x) :) = 0$
- (A4) On Minkowski spacetime and in the Minkowski vacuum state, $\omega_{Mink}(: T_{\mu\nu}(x) :) = 0$.
- (A5) $\omega(: T_{\mu\nu}(x) :)$ does not contain derivatives of the metric of order higher than two.

Uniqueness of Wald's $\omega(:T_{\mu\nu}(x):)$

- Any $\omega(:T_{\mu\nu}(x):)$ fulfilling the five axioms is unique up to a conserved local curvature term that vanishes in locally flat regions of M. [Wald]

- Requiring appropriate scaling and analyticity in m [Hollands & Wald]: the only sensible choices are $m^2 G_{\mu\nu}$ and

$$
\begin{aligned}
I_{\mu\nu} &\doteq \frac{1}{\sqrt{|g|}} \frac{\delta}{\delta g_{\mu\nu}} \int_M R^2 d\mu_g \\
&= g_{\mu\nu}\left(\frac{1}{2}R^2 - 2\Box R\right) + 2R_{;\mu\nu} - 2RR_{\mu\nu} \\
J_{\mu\nu} &\doteq \frac{1}{\sqrt{|g|}} \frac{\delta}{\delta g_{\mu\nu}} \int_M R_{\rho\tau}R^{\rho\tau} d\mu_g \\
&= \frac{1}{2} g_{\mu\nu}(R_{\mu\nu}R^{\mu\nu} - \Box R) + R_{;\mu\nu} - \Box R_{\mu\nu} - 2R_{\rho\tau}R^{\rho}{}_{\mu}{}^{\tau}{}_{\nu}.
\end{aligned}
$$

Which $D_{\mu\nu}$?

- $D'_y H$ does not satisfy the Dirac equations, thus $D^{can}_{\mu\nu}$ yields neither a conserved nor a traceless $\omega(:T_{\mu\nu}(x):)$.
- Possible solution: Add multiples of $L(\psi)$ to $T_{\mu\nu}$.
- This amounts to the choice

$$D^{c}_{\mu\nu} \doteq D^{can}_{\mu\nu} - \frac{c}{2} g_{\mu\nu} \left(D'_x + D_y\right) D'_y .$$

- It turns out that one can not assure both conservation and vanishing trace in the conformally invariant case!

The winner is $c = -\frac{1}{6}$.

Theorem

Let $\lambda_m \doteq 2\exp(\frac{7}{2} - 2\gamma)m^{-2}$ for $m \neq 0$ and λ_m arbitrary for $m = 0$, where γ denotes the Euler-Mascheroni constant, fix $\lambda = \lambda_m$ in the definition of H and let $\omega(: T_{ab}(x) :)$ be defined as discussed with $D_{\mu\nu} = D_{\mu\nu}^{-1/6}$ defined as above. Then $\omega(: T_{ab}(x) :)$ fulfills the first four of Wald's axioms. Furthermore, it exhibits the following trace anomaly

$$g^{\mu\nu}\omega(: T_{\mu\nu}(x) :) =$$

$$\frac{1}{\pi^2}\left(\frac{1}{1152}R^2 - \frac{1}{480}\Box R - \frac{1}{720}R_{\mu\nu}R^{\mu\nu} - \frac{7}{5760}R_{\mu\nu\rho\tau}R^{\mu\nu\rho\tau}\right).$$

Sketch of the proof

- Leaving c unspecified, one computes

$$8\pi^2 \nabla^\mu \omega(:T_{\mu\nu}(x):) = (1+6c)\, Tr[V_1(x,y)]_{;\nu}$$

and $$8\pi^2 g^{\mu\nu} \omega(:T_{\mu\nu}(x):) = 6(4c+1)\, Tr[V_1(x,y)] + m Tr\left[D'_y W^-(x,y)\right].$$

This gives (A3) and the trace anomaly.

- (A1) holds for Hadamard states ω, since adding multiples of $L(\psi)$ to $T_{\mu\nu}$ amounts to adding multiples of $Tr[V_1]$ to $\omega(:T_{\mu\nu}(x):)$.

- (A2) holds since $\omega(:T_{\mu\nu}(x):)$ is constructed entirely out of ω^- and H; these are preserved by χ.

- (A4) follows by straightforward computation.

Comments

- Scalar fields: Similar results are available. [Moretti]
- Dirac fields: Trace anomaly has already been computed, though based on a non-rigorous "heat-kernel-expansion". [Christensen & Duff]
- $\lambda \to \lambda' \quad \Rightarrow \quad \omega(:T_{\mu\nu}(x):)$ changes by multiples of

$$Tr[D_{\mu\nu}^{-\frac{1}{6}} V] = \frac{m^4}{2} g_{\mu\nu} - \frac{m^2}{6} G_{\mu\nu} + \frac{1}{60} (I_{\mu\nu} - 3J_{\mu\nu})$$

- Assuring (A5) therefore seems impossible for $m = 0$, but is possible for the trace.
- Different point of view: Defining both $:T_{\mu\nu}(x):$ and $:\nabla^\mu T_{\mu\nu}(x):$ as locally covariant quantum fields and using the renormalisation freedom (via further requirements) to assure $:\nabla^\mu T_{\mu\nu}(x): \equiv 0$. [Hollands & Wald]

Conclusions & Outlook

Conclusions & Outlook

- We have been able to define an (almost) sensible sourceterm for the semiclassical Einstein equation.
- In Robertson-Walker spacetimes one can [Dappiaggi, Fredenhagen, Pinamonti]
  1. re-express $G_{\mu\nu}(x) = 8\pi G\omega(: T_{\mu\nu}(x) :)$ as an equation for the traces
  2. and obtain solutions, stable at late times, which offer a potential description of "dark energy".
  3. How do these solutions look like for interacting fields?
- Maybe one can fulfil (A5) in the general case for special states?

Thank you for your attention!