Remarks on the expected stress-energy tensor of quantized Dirac fields on curved backgrounds

Thomas Hack

II. Institut für Theoretische Physik
Universität Hamburg

joint work with C. Dappiaggi & N. Pinamonti
part of PhD thesis under supervision of Prof. Fredenhagen

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Why QFT on CST?

- A quantum theory of both matter and gravity is not available yet.
- Partial solution: Quantum Field Theory on Curved Spacetimes (QFT on CST)
- This arises presumably as (some) semiclassical limit of (some) full quantum theory of gravity and matter ...
- ... and hopefully offers glimpses into features of the full theory (e.g. black Hole thermodynamics, cosmology).

Why Dirac fields?

- Most of the work in rigorous QFT on CST has considered scalar fields, since ...
- ... the simplest model was believed to already unravel the most difficult conceptual issues
- ... and it is in the nature of mathematical physicists to fear indices.
- We now understand why!
- Alas, the standard model contains only one (unobserved) scalar field, therefore higher spins need to be considered.

Why should you listen to this talk?

- QFT on CST is in principle formulated on an arbitrary but given background.
- The back-reaction of the quantum field on the spacetime can be described by
$$G_{\mu\nu} = 8\pi G \omega(:T_{\mu\nu}:) \quad (1).$$
- This can be formally derived by expanding around a vacuum solution, keeping "one-loop" ($\hbar^1$) terms of the quantum matter and "tree" ($\hbar^0$) terms of the quantum metric ...
- ... and can thus only make sense for special states or as a model equation.
- Taking it as starting point, which $:T_{\mu\nu}:$ and which ω should one use to obtain a meaningful r.h.s. for (1) (in the case of Dirac fields)?

Outline of the talk

1. Classical Dirac fields on curved spacetimes
2. Quantisation in the framework of AQFT
3. Hadamard states
4. The expected stress-energy tensor
5. Conclusions

The Dirac field on Minkowski spacetime

- We have global Poincaré invariance.
- The Dirac field is a map $\mathbb{R}^{3,1} \longrightarrow \mathbb{C}^4$ which transforms under the $D^{(\frac{1}{2},\frac{1}{2})}$ representation of $SL(2,\mathbb{C}) \simeq Spin^0(3,1)$, the double cover of $\mathcal{L}_+^\uparrow = SO^0(3,1)$, once a Poincaré transformation is employed.
- Transformation properties determine unambiguously the dynamics, i.e. the Dirac equation.

The Dirac field on curved spacetimes

- In this talk: A spacetime (M, g) is a fourdimensional, Hausdorff, globally hyperbolic, smooth manifold M with smooth Lorentzian metric g of signature $(-,+,+,+)$.

- We have only local Lorentz invariance, we thus can only hope to

  1. describe the Dirac field ψ as a section of a $\mathbb{C}^4$-bundle ($\longrightarrow$ Dirac bundles),
  2. assure a globally consistent local double-covering $Spin^0(3,1) \twoheadrightarrow \mathcal{L}_+^\uparrow$ to define sensible transformation properties of ψ ($\longrightarrow$ spin structure)
  3. and take the generally covariant generalisation of the Minkowskian Dirac equation ($\longrightarrow$ spin connection, γ-matrices).

Gamma matrices

- The Dirac algebra $Cl(3,1)$ is the Clifford algebra of $\mathbb{R}^{3,1}$ generated by $\mathbf{1}$ and the ONB b_a of $\mathbb{R}^{3,1}$ subject to

$$b_a b_b + b_b b_a = 2\eta_{ab}\mathbf{1}.$$

- Different complex irreducible representations $\pi^{(\prime)} : Cl(3,1) \to M(4,\mathbb{C})$ of $Cl(3,1)$ are equivalent by $\pi^{'}(c) = K\pi(c)K^{-1}$, $K \in GL(4,\mathbb{C})$. (Pauli)

- Dirac and charge conjugation matrices β_π, $\mathcal{C}_\pi$:

$$\beta_\pi^* = \beta_\pi, \quad i\beta_\pi\pi(n) > 0 \text{ for } n \text{ future pointing \& timelike,}$$
$$\pi(b_a)^* = -\beta_\pi\pi(b_a)\beta_\pi^{-1}, \quad \beta_\pi \text{ unique up to } \mathbb{R}_+,$$
$$\overline{\mathcal{C}_\pi}\mathcal{C}_\pi = \mathbf{1}, \quad \overline{\pi(b_a)} = \mathcal{C}_\pi\pi(b_a)\mathcal{C}_\pi^{-1}, \quad \mathcal{C}_\pi \text{ unique up to } U(1)$$

- Requiring that the relations of β_π and $\mathcal{C}_\pi$ are preserved in a change of representation determines K up to a sign.

- In the following we choose an arbitrary but fixed π and define $\gamma_a \doteq \pi(b_a)$.

The spin group

- One defines

$$Pin(3,1) \doteq \left\{ c \in Cl(3,1) \,|\, c = u_1 \cdots u_k, \quad k \in \mathbb{N}, \quad u_i \in \mathbb{R}^{3,1}, \quad u_i^2 = \pm \mathbf{1} \right\},$$

$$Spin(3,1) \doteq Pin(3,1) \cap Cl^0(3,1),$$

$$Spin^0(3,1) \doteq \text{connected component of } Spin(3,1) \ni \mathbf{1}.$$

- One can show

1. $\Lambda : Pin(3,1) \to \mathcal{L} = O(3,1)$ defined as

$$S b_a S^{-1} = b_b \Lambda^b{}_a(S)$$

is a double covering homomorphism and restricts to a double covering homomorphism $Spin^0(3,1) \twoheadrightarrow \mathcal{L}_+^\uparrow$.

2. $d\Lambda : \mathfrak{spin}^0(3,1) \to \mathfrak{l}_+^\uparrow$, the derivative of Λ at the identity fulfills

$$(d\Lambda)^{-1}(\lambda^a{}_b) = \frac{1}{4} \lambda^{ab} b_a b_b.$$

Spin structures I

- In the following: Let (M, g) be oriented and time-oriented.
- The Lorentz frame bundle $LM(\mathcal{L}_+^\uparrow, \pi_L, R_L, M)$ is a principle $\mathcal{L}_+^\uparrow$-bundle over M, which can be thought of the collection of all orthogonal frames $\{e_a\}_{a=0..3}$, $g(e_a, e_b) = \eta_{ab}$ at every point in M.
- A spin structure on (M, g) is a pair (SM, ρ), where $SM(Spin^0(3,1), \pi_S, M, R_S)$ is a principle $Spin^0(3,1)$-bundle over M and ρ is a diffeomorphism $SM \to LM$ s.t.

$$\rho \circ \pi_S = \pi_L \circ \rho \quad \text{and} \quad \rho \circ R_S(S) = R_L(\Lambda(S)) \circ \rho.$$

- Two spin structures (SM_1, ρ_1), (SM_2, ρ_2) are called equivalent if there exists a base point preserving isomorphism $\tau : SM_1 \to SM_2$ s.t. $\rho_2 \circ \tau = \rho_1$.

Spin structures II

- Existence & uniqueness (up to equivalence) of spin structures:

1. A spin structure exists iff the second Stiefel-Whitney class $w_2(M) \in \check{H}^2(M, \mathbb{Z}_2)$ is trivial. (Borel, Hirzebruch)

2. A spin structure exists iff the first fundamental group of LM splits as

$$\pi_1(LM) \simeq \pi_1(M) \times \pi_1(\mathcal{L}_+^\uparrow) \simeq \pi_1(M) \times \mathbb{Z}_2. \text{ (Geroch)}$$

3. For non-compact M: A spin structure exists iff M is parallelisable, i.e. LM can be globally trivialised. (Geroch)

4. $\Rightarrow$ On fourdimensional, globally hyperbolic spacetimes (M, g) spin structures always exist! (Geroch)

5. A spin structure is unique up to equivalence iff $\pi_1(M)$ is trivial. (Milnor)

Dirac fields I

- The Dirac bundle DM is the associated vector bundle $DM \doteq SM \times_\pi \mathbb{C}^4$.
- The dual Dirac bundle D^*M can be defined straightforwardly employing the hermitian inner product on $\mathbb{C}^4$.
- We define the spaces of smooth sections $\Gamma(M, DM)$, $\Gamma(M, D^*M)$ and smooth sections with compact support $\mathcal{D}(M, DM)$, $\mathcal{D}(M, D^*M)$.
- Dirac spinor fields ψ are elements of $\Gamma(M, DM)$ understood as column vectors.
- Dirac cospinor fields ψ' are elements of $\Gamma(M, D^*M)$ understood as row vectors.

Dirac fields II

- We define a global pairing of $\mathcal{D}(M, DM)$ and $\Gamma(M, D^*M)$ or $\Gamma(M, DM)$ and $\mathcal{D}(M, D^*M)$ as

$$\langle \psi' \psi \rangle \doteq \int_M d\mu_g(x)\, \psi'(x)\psi(x).$$

- We define maps

$$^\dagger : \Gamma(M, DM) \to \Gamma(M, D^*M), \quad ^\dagger : \Gamma(M, D^*M) \to \Gamma(M, DM),$$
$$^c : \Gamma(M, DM) \to \Gamma(M, DM), \quad ^c : \Gamma(M, D^*M) \to \Gamma(M, D^*M)$$

as

$$\psi^\dagger(x) \doteq \psi^{*\dagger}(x)\beta_\pi, \quad \psi'^\dagger(x) \doteq \beta_\pi^{-1}\psi'^*(x),$$
$$\psi^c(x) \doteq \mathcal{C}_\pi^{-1}\overline{\psi(x)}, \quad \psi'^c(x) \doteq \overline{\psi'^c(x)}\mathcal{C}_\pi.$$

Spinor-tensors

- Spinor-tensors are sections on mixed (inner) tensor products of TM, T^*M, DM and D^*M. On fourdimensional, globally hyperbolic (M,g) these bundles are all trivial!

- We can choose thus global orthogonal frames

$$E \doteq \{E_A\}_{A=1..4}, \quad \{E^B\}_{B=1..4}, \quad E^B E_A = \delta^B_A,$$
$$E_A \in \Gamma(M,DM), \quad E^B \in \Gamma(M,D^*M), \quad E \in \Gamma(M,SM),$$
$$e \doteq \{e_a\}_{a=0..3}, \quad \{e^b\}_{b=0..3}, \quad g(e^b,e_a) = \delta^a_b,$$
$$e_a \in \Gamma(M,TM), \quad e^b \in \Gamma(M,T^*M), \quad e \in \Gamma(M,LM), \quad e = \rho \circ E$$

  and express spinor-tensors in mixed tensor products of these frames.

- Example: The section of γ-matrices $\gamma \doteq \gamma^A_{aB}\, e^a \otimes E_A \otimes E^B$.

- One can switch from the frame basis e_a, e^b to a coordinate basis ∂_μ, d^ν via contraction with the coefficients e^μ_a, e^a_ν of the frame basis, e.g. $\gamma^A_{\mu B} \doteq e^a_\mu \gamma^A_{aB}$.

The spin connection

- Let $\mathcal{G}$ denote the $\mathfrak{l}_+^{\uparrow}$-valued connection form of the Levi-Civita connection on (M, g). One can then define the connection form Σ of the spin connection as
$$\Sigma \doteq (d\Lambda)^{-1} \circ \rho^*(\Gamma).$$
- The connection coefficients are
$$\Gamma^a_{bc} \doteq \mathcal{G}^a{}_c \circ e^*(e_b), \quad \sigma^A_{aB} \doteq \Sigma^A{}_B \circ E^*(e_a) = \frac{1}{4}\Gamma^a_{bc}(\gamma_a\gamma^c)^A{}_B.$$
- Equivalently: $\Gamma^a_{bc} = g(e^a, \nabla_{e_b} e_c)$ and $\sigma^A_{aB} = E^A \nabla_{e_b} E_B$.
- We can thus define covariant derivatives on spinor-tensors.
- Example:
$$\nabla_{e_a}\gamma^A_{bB} \doteq \nabla_a \gamma^A_{bB} \doteq \gamma^A_{bB;a} = \partial_a \gamma^A_{bB} + \sigma^A_{aC}\gamma^C_{bB} - \sigma^C_{aB}\gamma^A_{bC} - \Gamma^c_{ab}\gamma^A_{cB} = 0$$

Curvature tensors

- The Riemann- and Ricci tensor as well as the Ricci scalar are defined via their components as

$$v_{a;cb} - v_{a;bc} \doteq R_{a\ bc}^{\ \lambda} v_\lambda , \qquad R_{ab} \doteq R_{a\ bd}^{\ d} , \qquad R \doteq R^a_{\ a} ,$$

where v_a are the components of an arbitrary covector.

- The curvature tensor C of the spin connection is defined as

$$V_{A;cb} - V_{A;bc} \doteq C_{A\ bc}^{\ B} V_B ,$$

where V_A are the components of an arbitrary cospinor.

- It follows that

$$C^A_{\ Bab} = \frac{1}{4} R_{abcd} \gamma^{cA}_{\quad C} \gamma^{dC}_{\quad B} .$$

The Dirac equations

- In the following Feynman slash notation $\not{v} \doteq v^a\gamma_a = v_a\gamma^a$ is employed and spinor indices are suppressed in most cases.
- We define the Dirac operators

$$D : \Gamma(M, DM) \to \Gamma(M, DM), \quad D : \Gamma(M, D^*M) \to \Gamma(M, D^*M),$$
$$D' : \Gamma(M, DM) \to \Gamma(M, DM), \quad D' : \Gamma(M, D^*M) \to \Gamma(M, D^*M)$$
$$\text{as} \quad D \doteq -\not{\nabla} + m \quad \text{and} \quad D' \doteq \not{\nabla} + m.$$

- $\psi \in \Gamma(M, DM)$, $\psi' \in \Gamma(M, D^*M)$ are said to satisfy the Dirac equations if

$$D\psi = -\gamma^a\nabla_a\psi + m\psi = 0 \quad \text{and} \quad D'\psi' = \nabla_a\psi'\gamma^a + m\psi = 0.$$

- It follows:
  1. † and ^{c} preserve the Dirac equations.
  2. Solutions of the Dirac equation are also solutions of the spinorial Klein-Gordon equation with the spinorial Klein-Gordon operator

$$P \doteq -D'D = -DD' = \nabla_a\nabla^a - \frac{R}{4} - m^2 \doteq \Box - \frac{R}{4} - m^2. \text{ (Lichnerowicz)}$$

Algebraic Quantum Field Theory

- One seeks to define an increasing net of C^*-algebras $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O}\subset M}$ s.t.
  1. $\mathcal{A}(\mathcal{O})$ represents the physical observables localised in $\mathcal{O}$,
  2. $\mathcal{A}(M) \doteq \overline{\bigcup_{\mathcal{O}\subset M}\mathcal{A}(\mathcal{O})}$ represents all physical observables in M,
  3. $\mathcal{O}\subset\mathcal{O}' \Rightarrow \mathcal{A}(\mathcal{O})\subset\mathcal{A}(\mathcal{O}')$,
  4. $[\mathcal{A}(\mathcal{O}),\mathcal{A}(\mathcal{O}')] = 0$ if $\mathcal{O}$ and $\mathcal{O}'$ are spacelike separated
  5. and other properties I would like to skip.
- A state ω is a positive, normed linear functional on $\mathcal{A}(M)$, i.e.

$$\omega(A^*A) \geq 0 \;\forall A\in\mathcal{A}(M), \quad \omega(\mathbf{1}) = 1.$$

- The relation to the Hilbert space formalism is provided by the *GNS*-representation, s.t. ω is represented as a "vacuum" vector and elements of $\mathcal{A}(M)$ as bounded linear operators.
- To construct $\mathcal{A}(\mathcal{O})$ we follow Dimock whose starting point are "equal-time" canonical anticommutation relations (CAR) on an arbitrary Cauchy surface.

The fundamental solutions of D and D'

- Dimock showed that there exist fundamental solutions of D and D', i.e.

  1. there exist unique maps $S^{\pm} : \mathcal{D}(M, DM) \to \Gamma(M, DM)$, $S^{\pm}_{*} : \mathcal{D}(M, D^{*}M) \to \Gamma(M, D^{*}M)$ which fulfil

  $$DS^{\pm} = S^{\pm}D = id_{\mathcal{D}(M,DM)}, \quad D'S^{\pm}_{*} = S^{\pm}_{*}D' = id_{\mathcal{D}(M,D^{*}M)},$$
  $$\text{supp}(S^{\pm}u) \subset J^{\pm}(\text{supp } u) \, \forall u \in \mathcal{D}(M, DM),$$
  $$\text{supp}(S^{\pm}_{*}v) \subset J^{\pm}(\text{supp } v) \, \forall v \in \mathcal{D}(M, D^{*}M).$$

  2. They can be specified as $S^{\pm} = D'E^{\pm}$ and $S^{\pm}_{*} = DE^{\pm}_{*}$, where $E^{\pm}$ and $E^{\pm}_{*}$ are the fundamental solutions of $-P$, which exist and are unique since P has metric principal symbol $g^{\mu\nu}\partial_{\mu}\partial_{\nu}$.

  3. The causal propagators $S \doteq S^{+} - S^{-}$ and $S_{*} \doteq S^{+}_{*} - S^{-}_{*}$ map compactly supported sections into solutions of the Dirac equations.

Dirac fields on a Cauchy surface

- Let Σ be an arbitrary smooth Cauchy surface of M and $\tilde{\varrho} : M \to \Sigma$ the restriction map.

- One can define the Dirac bundles on the Cauchy surface as pullback bundles $D\Sigma \doteq \varrho^*(DM)$, $D^*\Sigma \doteq \varrho^*(D^*M)$ and introduce the function spaces $\Gamma(\Sigma, D\Sigma)$, $\Gamma(\Sigma, D^*\Sigma)$, $\mathcal{D}(\Sigma, D\Sigma)$, $\mathcal{D}(\Sigma, D^*\Sigma)$ as well as the dual pairing of suitable sections $\langle \cdot \cdot \rangle$ in obvious notation.

- $\tilde{\varrho}$ induces restriction maps $\varrho : \Gamma(M, DM) \to \Gamma(\Sigma, D\Sigma)$ and $\varrho^* : \Gamma(M, D^*M) \to \Gamma(\Sigma, D^*\Sigma)$.

- There are positive definite sesquilinear products on $\mathcal{D}(\Sigma, D\Sigma)$ and $\mathcal{D}(\Sigma, D^*\Sigma)$, namely $(u_1, u_2) \doteq i\langle u_1^\dagger \not{n} u_2\rangle$ and $(v_1, v_2) \doteq -i\langle v_1 \not{n} v_2^\dagger\rangle$, where n is the forward pointing unit normal of Σ.

- We can thus define two Hilbert spaces by completion w.r.t. $(\cdot, \cdot)$, $\mathcal{H}_\Sigma \doteq \overline{\mathcal{D}(\Sigma, D\Sigma)}$ and $\mathcal{H}^*_\Sigma \doteq \overline{\mathcal{D}(\Sigma, D^*\Sigma)}$ and these are dual to each other.

The CAR algebra of fields

- We call a representation of the CAR over Σ a choice of Hilbert space $\mathcal{H}$ and two continuous linear maps

$$\chi : \mathcal{H}^*_\Sigma \to \mathcal{BL}(\mathcal{H}), \quad \text{and} \chi_* : \mathcal{H}_\Sigma \to \mathcal{BL}(\mathcal{H}),$$
$$\text{s.t. } \forall u \in \mathcal{H}_\Sigma,\ v \in \mathcal{H}^*_\Sigma :$$
$$\chi_*(u) = [\chi((u, \cdot))]^* \quad \text{and} \quad \{\chi(v), \chi_*(u)\} = \langle vu \rangle,$$

  where all other anticommutators are set to vanish.

- For $f \in \mathcal{D}(M, DM)$, $g \in \mathcal{D}(M, D^*M)$ we define

$$\psi(g) \doteq -\chi[\varrho_* S_*(g \not{n})] \quad \text{and} \quad \psi^\dagger(f) \doteq \psi(f^\dagger).$$

- The local field algebra $\mathcal{F}(\mathcal{O})$ is defined as the C^* -algebra generated by $\psi^\dagger(f)$, $\psi(g)$, where one employs all test sections f and g supported in $\mathcal{O}$.

The algebra of observables

- It follows that $\psi(g)$ and $\psi^\dagger(f)$ fulfil the distributional Dirac equations
$$D\psi(g) \doteq \psi(D'g) = 0 \quad \text{and} \quad D'\psi^\dagger(f) \doteq \psi^\dagger(Df) = 0 \ldots$$
- … and the CAR $\{\psi(g), \psi^\dagger(f)\} = \langle h(Sf)\rangle$.
- $\mathcal{F}(\mathcal{O})$ is thus not an algebra of observables!
- But we can define $\mathcal{A}(\mathcal{O})$ as the subset of $\mathcal{F}(\mathcal{O})$ generated by an even number of $\psi^\dagger(f)$ and $\psi(g)$.
- Local commutativity follows for spacelike separated $\mathcal{O}$ and $\mathcal{O}'$:
$$[A_\mathcal{O}, A'_\mathcal{O}] = 0, \; \forall A_\mathcal{O} \in \mathcal{A}(\mathcal{O}) \quad \text{and} \quad A'_\mathcal{O} \in \mathcal{A}(\mathcal{O}').$$

Uniqueness of $\mathcal{A}(\mathcal{O})$

- The construction of $\mathcal{A}(\mathcal{O})$ depends on
  1. a choice of spin structure (S, ρ) out of equivalent ones,
  2. an irreducible complex representation π of $Cl(3,1)$
  3. and a representation of the CAR over some Σ, i.e. a triple (Σ, χ, χ_*).
- How unique are the $\mathcal{A}(\mathcal{O})$? Answers can be formulated in a categorical framework.
- Dimock: Different choices of (S, ρ) and (Σ, χ, χ_*) give rise to functorially equivalent $\mathcal{A}(\mathcal{O})$.
- Sanders: $\mathcal{A}(\mathcal{O})$ can be constructed without (Σ, χ, χ_*) and different choices of (S, ρ) and π give rise to functorially equivalent $\mathcal{A}(\mathcal{O})$. $\psi^\dagger(f)$, $\psi(g)$ can be constructed as locally covariant fields in the sense of Brunetti, Fredenhagen and Verch.

Preferred states

- Minkowski: The isometry group (Poincaré group) and a spectrum condition fix a unique vacuum state.

- CST: The isometry group is in general trivial but a generalised spectrum condition can be formulated, the microlocal spectrum condition (μSC). This fixes only a class of states, the Hadamard states.

- It turns out that they have the same UV behaviour as the Minkowski vacuum (Radzikowski) and are furthermore quasiequivalent, which basically means that the density matrix states on their GNS-Hilbert spaces are equivalent (Verch).

- We will see that Hadamard states are well-suited for a definition of $\omega(: T_{\mu\nu} :)$!

Quasifree states

- "Preselection" of states for free fields: A state ω on $\mathcal{A}(M)$ is called quasifree if

$$\omega\left(\psi^\dagger(f_1)\cdots\psi^\dagger(f_m)\psi(g_1)\cdots\psi(g_n)\right)$$
$$=\delta_{mn}\sum_{\pi_m\in S_m}\prod_{i=1..m}\mathrm{sign}(\pi_m)\omega\left(\psi^\dagger(f_i)\psi(g_{\pi_m(i)})\right).$$

- Motivation: The GNS-Hilbert spaces of quasifree states are unitarily equivalent to Fock spaces.

- We restrict to quasifree states and analyse

$$\omega^+(f,g)\doteq\omega\left(\psi(g)\psi^\dagger(f)\right)\quad\text{and}\quad\omega^-(f,g)\doteq\omega\left(\psi^\dagger(f)\psi(g)\right).$$

Wavefront sets

- Wavefront sets (WF) specify singular "points" and singular "directions" of a distribution u' (e.g. $u' \in \mathcal{D}'(M, DM)$).
- WF are subsets of $T^*M \setminus \{0\}$, i.e. transform covariantly.
- The pointwise product of two distributions u'_1 and u'_2 is in general not well-defined, but only if

  $$WF(u'_1) \oplus WF(u'_2) \doteq \{(x, k_1 + k_2) \,|\, (x, k_1) \in \mathrm{WF}(u'_1),\, (x, k_2) \in \mathrm{WF}(u'_1)\}$$

  does not contain an element of the form $(x, 0)$.
- $\mathrm{WF}(pu') \subset \mathrm{WF}(u')$ and $\mathrm{WF}(fu') \subset \mathrm{WF}(u')$ for any partial differential operator p and any smooth function f.
- If $(\Box + f)u'$ is smooth than for any $(x, k) \in \mathrm{WF}(u')$ k is null and $\mathrm{WF}(u')$ contains the full null geodesic specified by the initial data (x, k) (propagation of singularities).

Bitensors

- Let VM (WN) be a vector bundle over M (N) with typical fibre V (W). The outer tensor product $VM \boxtimes WN$ is then a vector bundle over $M \times N$ with typical fibre $V \otimes W$.
- A bitensor is a section of an outer tensor product bundle. Example: $\omega^-(x, y)$ is a (distributional) section of $D^*M \boxtimes DM$.
- Primed indices denote components "at y", unprimed indices components "at x". Example: $\omega^-(x, y) = \omega^-_A{}^{B'}(x, y)\, E^A(x) \otimes E_{B'}(y)$.
- We will use Synge's bracket notation [ ] to denote the coinciding point limit of a smooth bitensor, e.g. $[B(x, y)_A{}^{B'}] \doteq B(x, x)_A{}^B$.
- $\sigma(x, y)$, $g(x, y)^{\mu}_{\nu'}$ and $I(x, y)^A{}_{B'}$ denote one half of the squared geodesic distance, the parallel transport of vectors and the one of spinors w.r.t. the geodesic connecting y to x. They are smooth on a geodesically convex neighbourhood. We define $\sigma_\mu \doteq \sigma_{;\mu}$.

The Hadamard form

- We say that $\omega^{\pm}(x,y)$ are of the Hadamard form if there exist smooth bispinors $U^{\pm}$, $V^{\pm}$ and $W^{\pm}$, s.t.

$$\omega^{\pm}(x,y) = \mp \frac{1}{8\pi^2} D'_y \left(H^{\pm}(x,y) + W^{\pm}(x,y) \right), \text{ where}$$

$$H^{\pm} \doteq \frac{U^{\pm}}{\sigma} + V^{\pm} \ln\left(\frac{\sigma}{\lambda^2}\right),$$

$$V^{\pm} = \sum_n V_n^{\pm} \sigma^n, \quad W^{\pm} = \sum_n W_n^{\pm} \sigma^n$$

  and λ denotes an arbitrary scale.

- The definition needs to be refined to avoid convergence problems of $V^{\pm}$ and $W^{\pm}$, obtain a well-defined distribution and rule out spacelike singularities. (Kay & Wald, Köhler)

The microlocal spectrum condition

- We say that $\omega^{\pm}(x,y)$ fulfil the μSC if

$$WF(\omega^{\pm}) = \left\{(x,k_x,y,k_y) \in (T^*M)^{\boxtimes 2} \setminus \{\mathbf{0}\}, \mid (x,k_x) \sim (y,k_y),\ k_x \triangleright (\triangleleft) 0\right\},$$

  where $\sim$ implies that it exists a null geodesic connecting x to y s.t. k_x (k_y) is its cotangent vector at x (y) and $k_x \triangleright 0$ ($k_x \triangleright 0$) denotes a future-directed (past-directed) covector.

- Hollands, Köhler, Kratzert, Sahlmann, Verch: $\omega^{\pm}(x,y)$ fulfil the μSC iff they are of Hadamard form.

- For states invariant under charge conjugation the conditions on $\omega^{\pm}(x,y)$ are related, in general they are not.

- In the following we will assume that $\omega^{\pm}(x,y)$ fulfil the μSC.

Determining the Hadamard coefficients

- Since $D'_x\omega^-(x,y) = D_y\omega^-(x,y) = 0$ we know that $D'_x D'_y H^-$ and $P_y H^-$ are smooth (but non-vanishing!) and similar statements hold for H^+.
- One can furthermore show

Proposition 1

$(D'_x - D_y)H^-$ and $P_x H^-$ are smooth.

- Employing these facts and the initial condition $[U^\pm] = \mathbf{1}$, one can show
  1. $U^- = uI^{-1}$ and $U^+ = uI$, where u is the Hadamard coefficient of the scalar Hadamard form.
  2. $V_n^\pm$ can be determined recursively out of $U^\pm$. $V^\pm$ is <u>not</u> proportional to the scalar Hadamard coefficient v.
  3. $U^\pm$ and $V^\pm$ depend only on the local curvature and m, while $W^\pm$ depends on the state ω.
- Calculations are greatly simplified if V^+ and V^- are "symmetric" and thus related, however there seems to be no proof yet. (work in progress)

Coincidence point limits of H^-

- For the analysis of $\omega(:T_{\mu\nu}:)$ we will need some conincidence point limits of derivatives of H^-

Proposition 2

The Hadamard bidistribution H^- fulfills

$$[V_1^-] = \left(\frac{m^4}{8} + \frac{m^2 R}{48} + \frac{R^2}{1152} - \frac{\Box R}{480} - \frac{R_{\mu\nu}R^{\mu\nu}}{720} + \frac{R_{\mu\nu\rho\tau}R^{\mu\nu\rho\tau}}{720}\right)\mathbf{1} + \frac{C_{\mu\nu}C^{\mu\nu}}{48}$$

$$[P_x H^-] = 6[V_1^-], \qquad [(P_x H^-)_{;\mu}] = 8[V_{1;\mu}], \qquad [(P_x H^-)_{;\mu'}] = -8[V_{1;\mu}] + 6[V_1^-]_{;\mu},$$

$$[P_y H^-] = 6[V_1^-], \qquad [(P_y H^-)_{;\mu}] = 8[V_{1;\mu}] - 2[V_1^-]_{;\mu}, \qquad [(P_y H^-)_{;\mu'}] = -8[V_{1;\mu}] + 8[V_1^-]_{;\mu},$$

$$Tr[D'_x D'_y H^-] = -Tr[P_x H^-], \qquad Tr[(D'_x D'_y H^-)_{;\mu}] = -Tr[(P_x H^-)_{;\mu}] + [V_1^-]_{;\mu},$$

$$Tr[(D'_x D'_y H^-)_{;\mu'}] = -Tr[(P_x H)_{;\mu'}] - [V_1^-]_{;\mu},$$

$$Tr[(P_y H^- - P_x H^-)_{;\nu'}]\gamma^\nu \gamma_\mu = 2Tr[V_1^-]_{;\mu}.$$

- Proof: seven months of calculations, to give you a flavor note that

$$[\sigma_{\alpha\beta\gamma\delta\varepsilon\phi\lambda}] = -\frac{1}{6} R_{\alpha\beta\gamma\delta;\varepsilon\phi\lambda} + 779 \text{ terms.}$$

The classical stress-energy tensor

- The Dirac equations for the classical ψ and its Dirac adjoint $\psi^\dagger$ can be realised as the minimum of the action functional

$$S[\psi] = \int_{M^4} d^4x \sqrt{|g|} L(\psi) = \int_{M^4} d^4x \sqrt{|g|} \left[\frac{1}{2} \psi^\dagger (D\psi) + \frac{1}{2} \left(D' \psi^\dagger \right) \psi \right]$$

- and the classical stress-energy tensor of Dirac fields is thus

$$T_{\mu\nu} \doteq \frac{1}{\sqrt{|g|}} \frac{\delta \sqrt{|g|} L(\psi)}{\delta g_{\mu\nu}} = \frac{1}{2} \left(\psi^\dagger \gamma_{(\mu} \psi_{;\nu)} - \psi^\dagger_{;(\mu} \gamma_{\nu)} \psi \right) - \frac{1}{2} L(\psi) g_{\mu\nu},$$

  where ( ) denotes symmetrisation and of course the Lagrangian vanishes on shell.

- Employing the Dirac equations, one can straightforwardly calculate

$$\nabla^\mu T_{\mu\nu} = 0 \quad \text{and} \quad g^{\mu\nu} T_{\mu\nu} = -m \psi^\dagger \psi,$$

  s.t. in particular the trace of $T_{\mu\nu}$ vanishes in the massless, i.e. conformally invariant, case.

Why are Hadamard states good for $\omega(:T_{\mu\nu}:)$?

- Let us recall $G_{\mu\nu}(x) = 8\pi G\omega(:T_{\mu\nu}(x):)$. This can only make sense if $\omega(:T_{\mu\nu}:)$ is finite and has finite fluctuations!
- One can in principle define Wick products like $:\psi^\dagger\psi(x):$ via normal ordering, i.e. subtraction of $\mp D'_y H^\pm$ and define $:T_{\mu\nu}(x):$ as a linear combination of Wick products.
- This would yield finite expectation values of $:T_{\mu\nu}(x):$ on all Hadamard states ω, since e.g.

$$\omega(:\psi^\dagger\psi(x):) = Tr[\omega^-(x,y) + D'_y H^-(x,y)] = -\frac{1}{8\pi^2} Tr[W^-(x,y)].$$

- It also yields finite fluctuations since a computation of these involves terms quartic in the field and thus terms like $\omega^+(x,y)\omega^-(y,x)$ appear. But these are well-defined if $\omega^\pm(x,y)$ fulfil the μSC!

Definition of $\omega(: T_{\mu\nu} :)$

- A satisfactory definition of Wick products is highly involved.
- We thus take an approach, which is equivalent but more straightforward: Subtract $-D'_y H^-$ from ω^-, take appropriate derivatives and then the coinciding point limit, s.t.

$$\omega(: T_{\mu\nu} :) \doteq \frac{1}{8\pi^2} Tr\left[D_{\mu\nu} W^-(x,y)\right],$$

  for some differential operator $D_{\mu\nu}$.
- Looking at the classical expression for $T_{\mu\nu}$, the canonical choice is

$$D^{can}_{\mu\nu} \doteq \frac{1}{2}\gamma_{(\mu}\left(\nabla_{\nu)} - g^{\nu'}_{\nu)}\nabla_{\nu'}\right) D'_y.$$

- But this choice turns out to be unsatisfactory!

Wald's axioms I

- To fix a "good" $D_{\mu\nu}$, we need to think about what we require from $\omega(: T_{\mu\nu} :)$. Wald has suggested five axioms.
- (A1) Given ω_1 and ω_2, such that $\omega_1^-(x,y) - \omega_2^-(x,y)$ is smooth,

$$\omega_1(: T_{\mu\nu}(x) :) - \omega_2(: T_{\mu\nu}(x) :) = Tr\left[\frac{1}{2}\gamma_{(\mu}\left(\nabla_{\nu)} - g_{\nu)}^{\nu'}\nabla_{\nu'}\right)\left(\omega_1^- - \omega_2^-\right)\right].$$

  ($\leftarrow$ The divergent part of $T_{\mu\nu}$ is proportional to $\mathbf{1}$.)

- (A2) $\omega(: T_{\mu\nu}(x) :)$ is local: Given two spacetimes (M, g) and (M', g') and two globally hyperbolic isometric neighbourhoods $x \in \mathcal{O} \subset M$ and $x' \in \mathcal{O}' \subset M'$, we can identify $\mathcal{A}(\mathcal{O})$ and $\mathcal{A}'(\mathcal{O}')$ by means of the isometry. If two states ω and ω' on $\mathcal{A}(M)$ and $\mathcal{A}'(M')$ coincide on $\mathcal{F}(\mathcal{O})$, then $\omega(: T_{\mu\nu}(x) :) = \omega'(: T_{\mu\nu}(x') :)$.

  ($\simeq$ The only non-local dependence of $\omega(: T_{\mu\nu}(x) :)$ is due to the state.)

Wald's axioms II

- (A3) $\nabla^\mu \omega(: T_{\mu\nu}(x) :) = 0$

  ($\leftarrow G_{\mu\nu}$ is conserved.)

- (A4) On Minkowski spacetime and in the Minkowski vacuum state $\omega_{Mink}(: T_{\mu\nu}(x) :) = 0$.

  ($\simeq \omega(: T_{\mu\nu}(x) :)$ is an extension of Minkowskian normal ordering.)

- (A5) $\omega(: T_{\mu\nu}(x) :)$ does not contain derivatives of the metric of order higher than two.

  ($\simeq$ Solutions of $G_{\mu\nu}(x) = 8\pi G \omega(: T_{\mu\nu}(x) :)$ should be stable.)

Uniqueness of Wald's $\omega(: T_{\mu\nu}(x) :)$

- Wald: Any $\omega(: T_{\mu\nu}(x) :)$ fulfilling the five axioms is unique up to a conserved local curvature term that vanishes in locally flat regions of M.

- Occam's razor: The only sensible choices are $m^2 G_{\mu\nu}$ and

$$
\begin{aligned}
I_{\mu\nu} &\doteq \frac{1}{\sqrt{|g|}} \frac{\delta}{\delta g_{\mu\nu}} \int_M R^2 d\mu_g \\
&= g_{\mu\nu} \left(\frac{1}{2} R^2 - 2\Box R \right) + 2R_{;\mu\nu} - 2RR_{\mu\nu} \\
J_{\mu\nu} &\doteq \frac{1}{\sqrt{|g|}} \frac{\delta}{\delta g_{\mu\nu}} \int_M R_{\rho\tau} R^{\rho\tau} d\mu_g \\
&= \frac{1}{2} g_{\mu\nu} (R_{\mu\nu} R^{\mu\nu} - \Box R) + R_{;\mu\nu} - \Box R_{\mu\nu} - 2R_{\rho\tau} R^{\rho}{}_{\mu}{}^{\tau}{}_{\nu}.
\end{aligned}
$$

Which $D_{\mu\nu}$?

- We regularise with $D'_y H^{\pm}$, which is not a solution of the Dirac equations. $\omega(:T_{\mu\nu}(x):)$ defined with $D^{can}_{\mu\nu}$ is thus neither conserved nor traceless in the conformally invariant case!
- Possible solution: Add multiples of the Lagrangian to $T_{\mu\nu}$. $L(\psi)$ vanishes on shell classically but maybe not on the quantum side, s.t. a potential "classical limit" is unchanged.
- This amounts to the choice

$$D^{c}_{\mu\nu} \doteq D^{can}_{\mu\nu} - \frac{c}{2} g_{\mu\nu} \left(D'_x + D_y\right) D'_y.$$

- I turns out that one can not assure both conservation and vanishing trace in the conformally invariant case!

The answer to all problems is $c = -\frac{1}{6}$.

Theorem

Let $\lambda_m \doteq 2\exp(\frac{7}{2} - 2\gamma)m^{-2}$ for $m \neq 0$ and λ_m arbitrary for $m = 0$, where γ denotes the Euler-Mascheroni constant, fix $\lambda = \lambda_m$ in the definition of H^- and let $\omega(: T_{ab}(x) :)$ be defined as discussed with $D_{\mu\nu} = D^{-1/6}_{\mu\nu}$ defined as above. Then $\omega(: T_{ab}(x) :)$ fulfills the first four of Wald's axioms. Furthermore, it exhibits the following trace anomaly

$$g^{\mu\nu}\omega(: T_{\mu\nu}(x) :) =$$

$$\frac{1}{\pi^2}\left(\frac{1}{1152}R^2 - \frac{1}{480}\Box R - \frac{1}{720}R_{\mu\nu}R^{\mu\nu} - \frac{7}{5760}R_{\mu\nu\rho\tau}R^{\mu\nu\rho\tau}\right).$$

Sketch of the proof

- Leaving c unspecified, one computes

$$8\pi^2 \nabla^\mu \omega(: T_{\mu\nu}(x) :) = (1+6c)\, Tr[V_1(x,y)]_{;\nu}$$

and $\quad 8\pi^2 g^{\mu\nu} \omega(: T_{\mu\nu}(x) :) = 6(4c+1)\, Tr[V_1(x,y)] + m\, Tr\left[D'_y W^-(x,y)\right].$

This gives (A3) and the trace anomaly.

- (A1) holds for a Hadamard state ω, since adding multiples of $L(\psi)$ to $T_{\mu\nu}$ amounts to adding multiples of $Tr[V1]$, i.e. state independent terms, to $\omega(: T_{\mu\nu}(x) :)$.

- (A2) holds since $\omega(: T_{\mu\nu}(x) :)$ is constructed entirely out of ω^- and H^- in a local manner.

- (A4) follows by straightforward computation.

The aftermath

- Moretti has obtained similar results along the same line for the scalar case.
- Christensen and Duff have obtained similar results for the Dirac field and higher spins, but with a "heat-kernel-expansion", which is not well-defined on general CST.
- We have been able to fix λ for $m > 0$. Changing λ amounts to a "redefinition" of W^- by multiples of V^-, s.t. $\omega(: T_{\mu\nu}(x) :)$ changes by multiples of

$$Tr[D^c_{\mu\nu} V] = \frac{m^4}{2} g_{\mu\nu} - \frac{m^2}{6} G_{\mu\nu} + \frac{1}{60} \left(I_{\mu\nu} - 3J_{\mu\nu} \right)$$

- This shows that it is sensible to restrict the renormalisation freedom to $G_{\mu\nu}$, $I_{\mu\nu}$ and $J_{\mu\nu}$ and that assuring (A5) seems impossible for $m = 0$.
- One can of course still add multiples of $G_{\mu\nu}$, $I_{\mu\nu}$ and $J_{\mu\nu}$ to $\omega(: T_{\mu\nu}(x) :)$ and understand this in the more general framework of locally covariant QFT, where Wick polynomials are already defined only up to local curvature terms. This approach has been pursued by Hollands and Wald.

Looking out for conclusions

- We have been able to define a sensible sourceterm for the semiclassical Einstein equation.
- In Robertson-Walker spacetimes one can re-express $G_{\mu\nu}(x) = 8\pi G\omega(:T_{\mu\nu}(x):)$ as an equation for the traces and a conservation equation, as done by Dappiaggi, Fredenhagen and Pinamonti. In this case (A5) can be fulfilled even for $m = 0$ and stable solutions can be obtained, which offer a potential description of "dark energy". Maybe one can fulfil (A5) in the general case for special states?
- What changes for interacting fields?

Thank you for your attention