Solution of the semiclassical Einstein equations with possible interpretation in cosmology[2]

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Plan of the Talk

- Introduction (Classical cosmological scenario)
- Semiclassical Einstein's equation
- Stress-Energy Tensor regularization
- Solution with scalar conformal fields as sources
- Solution with massive fields as sources

Cosmological scenario: geometry and matter

- **Physical input:** Universe is homogeneous and isotropic. Then FRW metric:

$$ds^2 = -dt^2 + a(t)^2 \left(\frac{dr^2}{1+\kappa r^2} + r^2 d\Sigma^2 \right).$$

$\kappa = 0$ flat, $\kappa = \pm 1$ open or closed.

- recent observation: $a(t) \simeq Ce^{Ht}$, and $\kappa \simeq 0$.
- Take a classical fluid for matter: $T_a{}^b = (-\rho, P, P, P)$
- Einstein's equations become FRW equations $H = \dot{a}a^{-1}$

$$3H^2 = 8\pi\rho - \frac{3\kappa}{a^2}, \qquad 3\dot{H} + 3H^2 = -4\pi\left(\rho + 3P\right)$$

- Eventually we shall use

$$-R = 8\pi T, \qquad \nabla^a T_{ab} = 0$$

it is equivalent up to an initial condition.

Cosmological scenario: observation

- If we use **Radiation**, **Dust** and **cosmological constant** to model the present day observations:
  - Radiation is less important. $\rho_R \sim a(t)^{-4}$
  - We look for a mixture of $\rho_M \sim a(t)^{-3}$ and $\rho_\Lambda \sim C$

We have a problem

in modeling CMB and Supernovae red-shift observation:

Total **Energy density** is:
$\sim$ **75%** *Cosmological constant*, $\sim$ **25%** *Dust*.

Known matter: only $\sim$ **4%**.

- Let's try to see the role of **quantum effects**.

Gravity: semiclassical approximation

- We would like to have a quantum theory of gravity.
- **Too difficult.**
- At least we would like to have a theory of backreaction.
- We try semiclassically.

$$G_{ab} = 8\pi \langle T_{ab} \rangle.$$

- It should work: when fluctuation of $\langle T_{ab} \rangle$ are negligible.
- As in atomic physics: quantum mechanical electron with external classical field.

Wald Axioms

In QM T_{ab} are singular objects $\langle T_{ab}\rangle \to \infty$.

We need a renormalisation prescription for T_{ab} on CST.

Wald axioms $\Longrightarrow$ meaningful semiclassical approx.
[Wald 77] [Wald 78]

(1.) It must agree with formal results for T_{ab}
(For scalar: $(\Phi, T_{ab}\Psi)$, can be found formally if $(\Phi, \Psi) = 0$).

(2.) Regularization of T_{ab} in Minkowski coincide with "normal ordering".

(3.) Conservation: $\nabla^a \langle T_{ab}\rangle = 0$.

(4.) Causality: $\langle T_{ab}\rangle$ at p depends only on $J^-(p)$.

(5.) T_{ab} depends on derivatives of the metric up to the second order (or third).

Matter: Scalar free field theory

- **Equation of motion:** We will consider $\xi = 1/6$.

$$P := -\Box + \xi R + m^2 \,, \qquad P\phi = 0 \,.$$

- **Stress-Energy Tensor:**

$$T_{ab} := \partial_a\phi\partial_b\phi - \frac{1}{6}g_{ab}\left(\partial_c\phi\partial^c\phi + m^2\phi^2\right) - \xi\nabla_{(a}\partial_{b)}\phi^2$$

$$+\xi\left(R_{ab} - \frac{R}{6}g_{ab}\right)\phi^2 + \left(\xi - \frac{1}{6}\right)g_{ab}\Box\phi^2.$$

- Conservation equations, and trace.

$$\nabla_a T^a{}_b = 0 \,, \qquad T = -3\left(\frac{1}{6} - \xi\right)\Box\phi^2 - m^2\phi^2 \,.$$

- It differs by the usual one by terms of the form:
  $g_{ab}\left(\phi P\phi + P\phi\phi\right)$ *[Moretti 2003]*.

Quantum field theory

- States in QFT are described by $n-$point functions.
- Quasi free states ω described by the two-points function

$$\omega_2(x,y) = \langle \phi(x)\phi(y) \rangle$$

thought as distribution in $\mathcal{D}'(M \times M)$.

- T_{ab} arises as an operation on ω_2 and a coinciding point limit.
- It is not well defined...
- Quasifree states that possess Hadamard property: *[Radzikowski 1995] [Brunetti Fredenhagen Köhler 1996]*
- **Physically:** The fluctuations of the field are always finite on Hadamard states.

Hadamard Two-points function

$$\omega_2 = \frac{1}{8\pi^2}\left(\frac{u}{\sigma_\epsilon} + v \log \sigma_\epsilon + w\right).$$

- σ is half of the square of the geodesic distance ,
- u v w are smooth functions,
- u depends only upon the geometry via g_{ab}
- v depends upon g_{ab}, ξ and m^2
- w characterizes the state.

The singular Structure $\mathbf{H} = \frac{1}{8\pi^2}\left(\frac{u}{\sigma} + v \log \sigma\right)$ is fixed and does not depend on the state.

Some notations:

$$v = \sum_{n=0}^{\infty} v_n \sigma^n \qquad [v](x) = v(x,x)$$

Later we will use $[v_1]$

Regularization of the two-points function

Regularization with point splitting: Minimal requirement.

$$\langle\phi(x)\phi(y)\rangle_\omega := \omega_2(x,y) - \mathbf{H}(x,y)$$

It reduces to normal ordering for flat spacetime.
T_{ab} build on it. *[Hollands Wald, Brunetti Fredenhagen Verch, Moretti]*

$$8\pi^2\langle\phi P\phi\rangle_\omega = 6[v_1], \qquad 8\pi^2\langle(\nabla_a\phi)(P\phi)\rangle_\omega = 2\nabla_a[v_1]$$

Conservation equation for T_{ab} are satisfied quantum mechanically

$$\nabla_a\langle T^a{}_b\rangle_\omega = 0$$

but (un)-fortunately the trace is different from the classical one.

$$\langle T\rangle_\omega := \frac{2[v_1]}{8\pi^2} + \left(-3\left(\frac{1}{6}-\xi\right)\Box - m^2\right)\frac{[w]}{8\pi^2}.$$

Some (long) computations.......

... or a look in the literature (for example *[Fulling]*) gives

$$2[v_1] = \frac{1}{360}\left(C_{ijkl}C^{ijkl} + R_{ij}R^{ij} - \frac{R^2}{3} + \Box R\right) + \frac{1}{4}\left(\frac{1}{6} - \xi\right)^2 R^2 +$$

$$+\frac{m^4}{4} - \frac{1}{2}\left(\frac{1}{6} - \xi\right) m^2 R + \frac{1}{12}\left(\frac{1}{6} - \xi\right)\Box R.$$

Remaining freedom

In the trace $c\Box R$. Wald's fifth axiom does not hold!

- **Other** regularization methods give different stress-energy tensors.
- **Difference:** We can add conserved tensors t_{ab} build out of the metric, m and ξ only.
- It must behave as $\langle T_{ab}\rangle$ under *"scale"* transformations.
- Some possibilities arises from the variation of

$$t_{ab} = \frac{\delta}{\delta g^{ab}} \int \sqrt{g} \left(C\ R^2 + D\ R_{ab}R^{ab} \right)$$

- The trace ${t_a}^a$ is proportional to $\Box R$
- We use this freedom to cancel the $\Box R$ term from $\langle T\rangle$.

Some Remarks:

- Wald's fifth axiom partially holds for $\langle T'_{ab}\rangle = \langle T_{ab}\rangle - t_{ab}$.
- **General principle of local covariance**: When regularization freedom is fixed in a region, is fixed in every spacetime. *[Brunetti Fredenhagen Verch 2003]*.
- The remaining freedom is $\langle\phi^2\rangle'_\omega = \langle\phi^2\rangle_\omega + A\ m^2 + B\ R$.
- But we can not completely cancel $[v_1]$ from $\langle T\rangle_\omega$.
- Similarities with $f(R)$ gravity, but t_{ab} alone does not guaranty stable solutions.

With $\kappa = 0$ and $\xi = 1/6$, the equation $-R = 8\pi\langle T\rangle$ becomes

$$-6\left(\dot{H} + 2H^2\right) = -8\pi G m^2 \langle\phi^2\rangle_\omega + \frac{G}{\pi}\left(-\frac{1}{30}\left(\dot{H}H^2 + H^4\right) + \frac{m^4}{4}\right)$$

Conformal invariant theory

If $\xi = \frac{1}{6}$, $m^2 = 0$, the equation does not depend on the state.

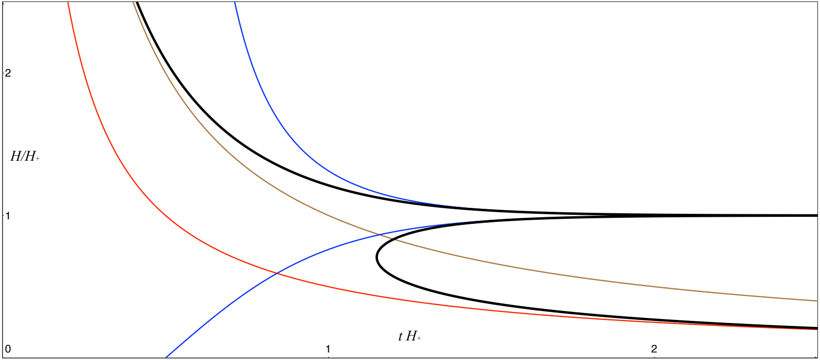
$$\dot{H}\left(H^2 - \frac{H_c^2}{2}\right) = -H^4 + H_c^2 H^2, \qquad H_c^2 = \frac{360\pi}{G}$$

$H^2 = H_c^2$ and $H^2 = 0$ are solutions *(de Sitter, Minkowksi)*.
They are both stable as seen by the full solution

$$Ce^{4t} = e^{2/H}\left|\frac{H + H_c}{H - H_c}\right|^{1/H_c}$$

- It is as in the Starobinsky model but now with stable de Sitter. *[Starobinsky 80, Vilenkin 85]*

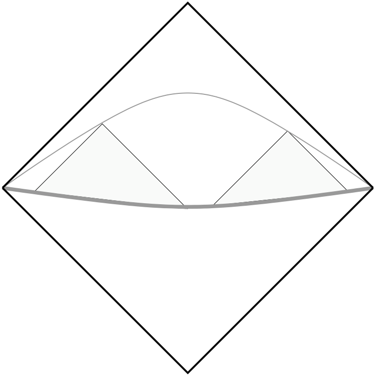
Clearly $H = 0$ and $H = H_c = H_+$ are stable solutions.



- $H = H_c$ is order of magnitude to big to describe the present expansion velocity of the universe.
- Two fixed points instead of one, a length scale is introduced (proportional to G).

Particle horizon

Maximal **comoving distance** (if $c = 1$) it is $\tau = \int_t^{t_1} \frac{dt}{a(t)}$.
Where is the singularity t_0 in the Penrose diagram?
Consider: $ds^2 = a^2 \left(-d\tau^2 + d\mathbf{x}^2\right)$.



- Radiation dominated:
  $\tau = \tau_0 - A(t - t_0)^{1/2} \to \tau_0$
  for $t \to t_0$
- Matter dominated:
  $\tau = \tau_0 - A(t - t_0)^{1/3} \to \tau_0$
  for $t \to t_0$
- $\rho = 1/a(t)^2$
  $\tau = \tau_0 - \log(t - t_0) \to -\infty$
  for $t \to t_0$

Massive model

Important: The quantum states enter in the equation via $\langle \phi^2 \rangle$.
We would like to use **"vacuum states"**.

Impossible. Adiabatic states, have similar properties.
[Parker, Parker and Fulling, Lüders Roberts, Junker Schrohe, Olbermann]

- Minimize the particle creation rate. *[Parker]*
- Minimal smeared energy in the sense of Fewster. *[Olbermann]*
- They can be thought as approximated ground states.

They are build in an approximated way by a sequence of ω_n.
We expand it in powers of $1/m^2$

$$\langle \phi^2 \rangle_{(n)} = Am^2 + BR + O\left(\frac{1}{m^2}\right)$$

The regime $m^2 >> R$ is what we need. If $m = 1 GeV$ $\frac{m^2}{R} \sim 10^{82}$

We have three parameters A, B, m.

$$\dot{H}\left(H^2 - H_0^2\right) = -H^4 + 2H_0^2 H^2 + M$$

where H_0 and M are two constants with the following values

$$H_0^2 = \frac{180\pi}{G} - 1440\pi^2 m^2 B, \qquad M = \frac{15}{2}m^4 - 240\pi^2 m^4 A$$

At most two fixed stable points (de Sitter phases)

$$H_\pm^2 = H_0^2 \pm \sqrt{H_0^4 + M}.$$

We want to have **Minkowski** $H_- = 0, \Longrightarrow A = (32\pi^2)^{-1}$.
Freedom in m and B to *"Fine tune"* H_+.

- H_+ can be made small by suitable choices of m^2 and A, B
- It could model dark energy.
- Quantum effects are hardly negligible.

Summary

- There is a regularization freedom not fixed by QFT.
- Semiclassical solutions of Einstein's equation fix in some sense the freedom.
- The solutions depend upon the quantum states.
- The de Sitter phases could be stable only fixing the renormalisation freedom.

Open Questions

- What happens considering more realistic models?
- Fluctuations?
- Connection with $f(R)$ gravity?
- Origin of R^2 terms in the action? An hint on quantum gravity?