Solutions of the semiclassical Einstein equations with possible interpretations in Cosmology [2]

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[2]Based on a joint work with C. Dappiaggi and K. Fredenhagen: arXiv:0801.2850 [gr-qc]

Plan of the Talk

- Introduction (Classical cosmological scenario)
- Semiclassical Einstein's equation
- Stress-Energy Tensor regularization
- Solution with scalar fields as sources

Cosmological scenario

- **Physical input:** Universe is homogeneous and isotropic.

$$ds^2 = -dt^2 + a(t)^2 \left(\frac{dr^2}{1+\kappa r^2} + r^2 d\Sigma^2 \right) .$$

- Dynamics of the Hubble parameter $H = \dot{a}a^{-1}$ is governed by

$$-R = 8\pi T, \qquad \nabla^a T_{ab} = 0.$$

- Classical fluid for matter: $T_a{}^b = (-\rho, P, P, P)$
- We use **Dust** and **cosmological constant** but....

We have a problem: To model observations.

Energy density: $\sim$ **75%** *Cosmological const.*, $\sim$ **25%** *Dust*.
Known matter: only $\sim$ **4%**.

- Let's try to see the role of **quantum effects**.

Gravity: semiclassical approximation

- We would like to have a quantum theory of gravity.
- **Too difficult.**
- At least we would like to have a theory of backreaction.
- We try semiclassically.

$$G_{ab} = 8\pi\langle T_{ab}\rangle.$$

- It should work: when fluctuation of $\langle T_{ab}\rangle$ are negligible.
- As in atomic physics: quantum mechanical electron with external classical field.

Wald Axioms

In QM T_{ab} are singular objects $\langle T_{ab}\rangle \to \infty$.

We need a renormalization prescription for T_{ab} on CST.

Wald axioms $\Longrightarrow$ meaningful semiclassical approx.
[Wald 77] [Wald 78]

(1.) It must agree with formal results for T_{ab}
(For scalar: $(\Phi, T_{ab}\Psi)$, can be found formally if $(\Phi, \Psi) = 0$).

(2.) Regularization of T_{ab} in Minkowski coincide with "normal ordering".

(3.) Conservation: $\nabla^a \langle T_{ab}\rangle = 0$.

(4.) Causality: $\langle T_{ab}\rangle$ at p depends only on $J^-(p)$.

(5.) T_{ab} depends on derivatives of the metric up to the second order (or third).

Quantum scalar field theory

- Equation of motion: $P := -\Box + \xi R + m^2 \,, \qquad P\phi = 0\,.$
- States in QFT are described by $n-$point functions.
- Quasi free states ω described by the two-points function

$$\omega_2(x,y) = \langle \phi(x)\phi(y) \rangle$$

thought as distribution in $\mathcal{D}'(M \times M)$.

- Quasifree states that possess Hadamard property:
[Radzikowski 1995] [Brunetti Fredenhagen Köhler 1996]

$$\mathrm{WF}(\omega_2) = \{((x_1,k_1),(x_2,k_2)) \in T^*M^2/\{0\} : -P_\gamma k_2 = k_1 > 0\}$$

- The singular structure $\mathbf{H} = \frac{u}{\sigma} + v \log \sigma$ is fixed depends only on g_{ab} and the potential.

Regularization of the two-points function

Regularization with point splitting: Minimal requirement.

$$\langle \phi(x)\phi(y)\rangle_\omega := \omega_2(x,y) - \mathbf{H}(x,y)$$

It reduces to normal ordering for flat spacetime.

$$T_{ab} := \partial_a\phi\partial_b\phi - \frac{1}{6}g_{ab}\left(\partial_c\phi\partial^c\phi + m^2\phi^2\right) - \xi\nabla_{(a}\partial_{b)}\phi^2 + \xi\left(R_{ab} - \frac{R}{6}g_{ab}\right)\phi^2 + \left(\xi - \frac{1}{6}\right)g_{ab}\Box\phi^2.$$

T_{ab} build on it. *[Moretti, Hollands Wald, Brunetti Fredenhagen Verch]*
Conservation equation for T_{ab} are satisfied quantum mechanically

$$\nabla_a\langle T^a{}_b\rangle_\omega = 0$$

but (un)-fortunately the trace is different from the classical one.

$$\langle T\rangle_\omega := \frac{V_1}{8\pi^2} + \left(-3\left(\frac{1}{6} - \xi\right)\Box - m^2\right)\frac{\langle\phi^2\rangle_\omega}{8\pi^2}.$$

Some (long) computations.......

... or a look in the literature (for example *[Fulling]*) gives

$$V_1 = \frac{1}{360}\left(C_{ijkl}C^{ijkl} + R_{ij}R^{ij} - \frac{R^2}{3} + \Box R\right) + \frac{1}{4}\left(\frac{1}{6} - \xi\right)^2 R^2 +$$

$$+\frac{m^4}{4} - \frac{1}{2}\left(\frac{1}{6} - \xi\right) m^2 R + \frac{1}{12}\left(\frac{1}{6} - \xi\right)\Box R.$$

Remaining freedom

In the trace $c\Box R$. Wald's fifth axiom does not hold!

- **Other** regularization methods give **different** $\langle T_{ab}\rangle$.
- **Difference:** Conserved tensors $t_{ab}(\xi, m, g)$.
- It must behave as $\langle T_{ab}\rangle$ under *"scale"* transformations.
- Some possibilities arises from the variation of

$$t_{ab} = \frac{\delta}{\delta g^{ab}} \int \sqrt{g} \left(C\ R^2 + D\ R_{ab}R^{ab} \right)$$

- The trace ${t_a}^a$ is proportional to $\Box R$
- We use this freedom to cancel the $\Box R$ term from $\langle T\rangle$.

Some Remarks:

- Wald's fifth axiom holds for the trace of $\langle T'_{ab}\rangle = \langle T_{ab}\rangle - t_{ab}$.
- **General principle of local covariance**: Fixed the freedom in a region then is fixed in every spacetime. *[Brunetti Fredenhagen Verch 2003] [Hollands Wald 2003]*.
- The remaining freedom is $\langle\phi^2\rangle'_\omega = \langle\phi^2\rangle_\omega + A\ m^2 + B\ R$.
- We can **not** completely cancel V_1 from $\langle T\rangle_\omega$.
- t_{ab} alone does not guaranty stable solutions.

With $\kappa = 0$ and $\xi = 1/6$, the equation $-R = 8\pi\langle T\rangle$ becomes

$$-6\left(\dot{H} + 2H^2\right) = -8\pi G m^2 \langle\phi^2\rangle_\omega + \frac{G}{\pi}\left(-\frac{1}{30}\left(\dot{H}H^2 + H^4\right) + \frac{m^4}{4}\right)$$

Scalar fields as sources

Adiabatic states, we expand it in powers of $1/m^2$

$$\langle\phi^2\rangle_{(n)} = Am^2 + BR + O\left(\frac{1}{m^2}\right)$$

Two parameter $H_0(m, B)$ and $M(m, A)$ depending on A, B, m in

$$\dot{H}\left(H^2 - H_0^2\right) = -H^4 + 2H_0^2 H^2 + M$$

At most two fixed stable points (de Sitter phases)

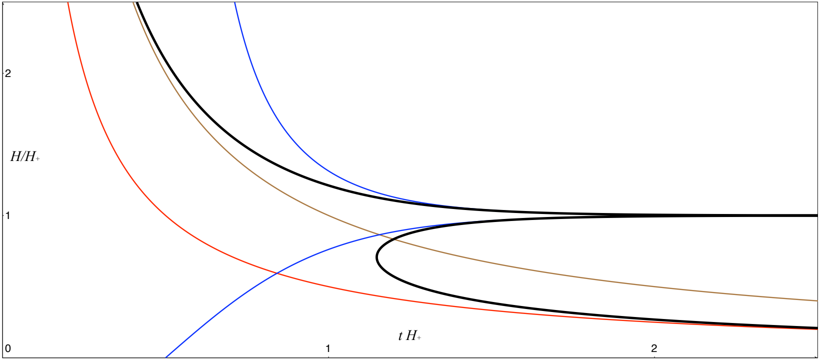
$$H_\pm^2 = H_0^2 \pm \sqrt{H_0^4 + M}.$$

We want to have **Minkowski** $H_- = 0, \implies M = 0.$

Freedom in m and B to *"Fine tune"* H_+.

$H(t) = 0$ and $H(t) = H_+$ are stable as seen by the solution

$$Ce^{4t} = e^{2/H} \left| \frac{H + H_+}{H - H_+} \right|^{1/H_+}$$

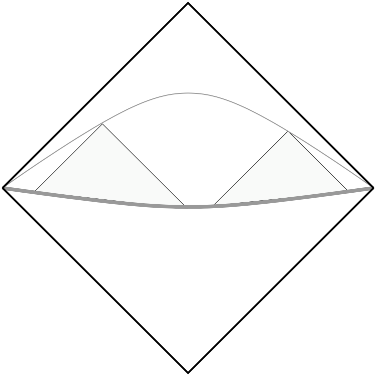


- It is similar to the Starobinsky model but now: 1) de Sitter is stable, 2) H_+ can be made small. *[Starobinsky 80, Vilenkin 85]*

Particle horizon

Maximal **comoving distance** (if $c = 1$) it is $\tau = \int_t^{t_1} \frac{dt}{a(t)}$.
Consider: $ds^2 = a^2 \left(-d\tau^2 + d\mathbf{x}^2\right)$.

Where is the singularity t_0 in the Penrose diagram?



- Radiation dominated:
  $\tau = \tau_0 - A(t - t_0)^{1/2} \to \tau_0$
  for $t \to t_0$
  Singularity is at finite time.

- $\rho = 1/a(t)^2$
  $\tau = \tau_0 - \log(t - t_0) \to -\infty$
  for $t \to t_0$
  Singularity is a null surface.

Summary

- There is a regularization freedom not fixed by QFT.
- Semiclassical solutions of Einstein's equation fix in some sense the freedom.
- The solutions depend upon the quantum states.
- The de Sitter phases could be stable only fixing the renormalization freedom.

Open Questions

- What happens considering more realistic models?
- Fluctuations?
- Connection with $f(R)$ gravity?
- Origin of R^2 terms in the action? An hint on quantum gravity?