Solutions of the semiclassical Einstein's equations with applications in cosmology[2]

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[2]Based on a joint work with C. Dappiaggi and K. Fredenhagen: arXiv:0801.2850 [gr-qc]

Summary

- Cosmological Scenario
- Semiclassical Einstein's equation
- Stress-Energy Tensor regularization
- Solution with scalar conformal fields as sources
- Solution with massive fields as sources

Cosmological scenario: geometry

- Physical input: Universe is homogeneous and isotropic.
  Then FRW metric:

$$ds^2 = -dt^2 + a(t)^2 \left(\frac{dr^2}{1+\kappa r^2} + r^2 d\Sigma^2 \right).$$

  $\kappa = 0$ flat, $\kappa = \pm 1$ open or closed.
- recent observation: $a(t) \simeq Ce^{Ht}$, and $\kappa \simeq 0$.

Solutions that have the FRW form.

Cosmological scenario: matter

- We model T_{ab} as a perfect fluid

$$T_{ab} = \rho u_a u_b + P(g_{ab} + u_a u_b).$$

Homogeneity and isotropy $\Longrightarrow u = \frac{\partial}{\partial t}$, $\rho(t)$ and $P(t)$

- Einstein's equations become FRW equations $H = \dot{a}a^{-1}$

$$3H^2 = 8\pi\rho - \frac{3\kappa}{a^2} \tag{1}$$

$$3\dot{H} + 3H^2 = -4\pi\left(\rho + 3P\right) \tag{2}$$

- Type of fluids: $P = w\rho$ and conservation equation
  - Radiation: $w = \frac{1}{3}$, $\rho_R \sim a(t)^{-4}$
  - Dust: $w = 0$, $\rho_M \sim a(t)^{-3}$
  - Cosmological constant: $w = -1$ $\rho_\Lambda = C$

Cosmological scenario: observation

- If we use this to model the present day observation:
  - Radiation is not important.
  - We look for a mixture of ρ_M and ρ_Λ
- To model CMB and Supernovae red-shift observation:

We have a problem

At the present time:
Energy density: $\sim$ 70% *Dark Energy*, $\sim$ 30% *Matter*.
Known matter: only $\sim$ 4%.

- Let's try to see the role of quantum effects.

Gravity: semiclassical approximation

- We would like to have a quantum theory of gravity.
- **Too difficult.**
- At least we would like to have a theory of backreaction.
- We try semiclassically.

$$G_{ab} = 8\pi \langle T_{ab} \rangle .$$

- It should work in some regimes. As in atomic physics: quantum mechanical electron with external classical field.

Range of validity of semiclassical approximation

- A complete satisfactory semiclassical description is impossible. (quantum matter is a source for gravity).
- It should be valid whenever quantum fluctuations are negligible.
- In some models, backreaction is unavoidable: Particle creation.
  (Ex: black holes radiates)
  - Are there quantum effects that can be seen?
  - How is modified the vacuum energy?
  - How can be treated the backreaction effects?
  - Are they a small effect?
  - What implication has the quantum origin of matter on the solutions?

Wald Axioms

In QM T_{ab} are singular objects $\langle T_{ab} \rangle \to \infty$.

We need a renormalization prescription for T_{ab} on CST.

Wald axioms $\Longrightarrow$ meaningful semiclassical approx.
[Wald 77] [Wald 78]

(1.) It must agree with formal results for T_{ab}
(For scalar: $(\Phi, T_{ab}\Psi)$, can be found formally if $(\Phi, \Psi) = 0$).

(2.) Regularization of T_{ab} in Minkowski coincide with "normal ordering".

(3.) Conservation: $\nabla^a \langle T_{ab} \rangle = 0$.

(4.) Causality: $\langle T_{ab} \rangle$ at p depends only on $J^-(p)$.

(5.) T_{ab} depends on derivatives of the metric up to the second order (or third).

Nice Environment

Problem:

How can we treat matter without fixing the spacetime?

- We can quantize simultaneously and coherently on all spacetimes. *[Brunetti Fredenhagen Verch 2003]*.
- Quantum Fields are particular observables that transform suitably under isomorphisms.
- We need another ingredient:
  "reference states" on every spacetime.
- Einstein's eq.
  - Consistency criterion.
  - Selects particular elements in the category of local Manifolds.

What we need to search for in a quantum theory

Instead of considering FRW equation we use the following.

$$-R = 8\pi T, \qquad \nabla^a T_{ab} = 0$$

- Up to some initial condition
  (it remains the freedom of fixing $a(t_0) = a_0$).
- But it is simpler to perform quantum computations.

Matter: Scalar free field theory

Equation of motion

$$P := -\Box + \xi R + V \,, \qquad P\phi = 0 \,.$$

We will be interested in the case $V = m^2$ and $\xi = 1/6$.
Stress-Energy Tensor:

$$T_{ab} := \partial_a\phi\partial_b\phi - \frac{1}{6}g_{ab}\left(\partial_c\phi\partial^c\phi + V\phi^2\right) - \xi\nabla_{(a}\partial_{b)}\phi^2$$

$$+\xi\left(R_{ab} - \frac{R}{6}g_{ab}\right)\phi^2 + \left(\xi - \frac{1}{6}\right)g_{ab}\Box\phi^2.$$

It differs by the usual one by terms of the form $\phi P\phi$ [Moretti 2003].

Remarks on T_{ab}

- Trace

$$T = -3\left(\frac{1}{6} - \xi\right)\Box\phi^2 - V\phi^2 \ ,$$

where we have used $\phi P \phi = 0$ to simplify.

- Conservation equation

$$\nabla_a T^a{}_b = -\frac{1}{2}\phi^2 \partial_b V \ .$$

- Classical Ambiguity: $\phi P \phi = 0$

$$T'_{ab} = T_{ab} + C g_{ab} \left(\phi P \phi + P\phi \phi\right) \ .$$

- T_{ab} can be written by means of balanced derivatives and derivatives of the field ϕ^2 *[Buchholz Ojima Roos 2002]*.

Quantum field theory

States become distribution,

- QFT described by $n-$point functions.
- Quasi free states ω described by the two-points function

$$\omega_2(x, y) = \langle \phi(x)\phi(y) \rangle$$

  thought as distribution in $\mathcal{D}'(M \times M)$.
- T_{ab} arises as an operation on ω_2 and a coinciding point limit.
- It is not well defined...
- Quasifree states that possess Hadamard property *[Radzikowski 1995] [Brunetti Fredenhagen Köhler 1996]*

$$\mathrm{WF}(\omega_2) = \{((x_1, k_1), (x_2, k_2)) \in T^*M^2/\{0\} : -P_\gamma k_2 = k_1 > 0\}$$

- **Physically:** The fluctuations of the field are always finite on Hadamard states.

Hadamard Two-points function

$$\omega_2 = \frac{1}{8\pi^2}\left(\frac{u}{\sigma_\epsilon} + v \log \sigma_\epsilon + w\right).$$

- u v w are smooth functions,
- u depends only upon the geometry via g_{ab}
- v depends upon g_{ab}, ξ and V
- w characterizes the state.

Some notations: σ is half of the square of the geodesic distance

$$v = \sum_{n=0}^{\infty} v_n \sigma^n \qquad [v](x) = v(x,x)$$

The singular Structure H is fixed and does not depend on the state.

Regularization of the two-points function

Regularization with point splitting: Minimal requirement.

$$\langle \phi(x)\phi(y)\rangle_\omega := \omega_2(x,y) - H(x,y)$$

It reduces to normal ordering for flat spacetime.
T_{ab} build on it. *[Hollands Wald, Brunetti Fredenhagen Verch, Moretti]*

$$8\pi^2\langle \phi P \phi\rangle_\omega = 6[v_1], \qquad 8\pi^2\langle (\nabla_a \phi)(P\phi)\rangle_\omega = 2\nabla_a[v_1]$$

Conservation equation for T_{ab} are satisfied quantum mechanically

$$\nabla_a \langle T^a{}_b\rangle_\omega = -\frac{1}{2}\langle \phi^2\rangle_\omega \partial_b V = -\frac{1}{2}\frac{[w]}{8\pi^2}\partial_b V$$

but unfortunately the trace is different from the classical one.

$$\langle T\rangle_\omega := \frac{2[v_1]}{8\pi^2} + \left(-3\left(\frac{1}{6}-\xi\right)\Box - V\right)\frac{[w]}{8\pi^2}.$$

Some computations.......

with $V = m^2$

$$2[v_1] = \frac{1}{360}\left(C_{ijkl}C^{ijkl} + R_{ij}R^{ij} - \frac{R^2}{3} + \Box R\right) + \frac{1}{4}\left(\frac{1}{6} - \xi\right)^2 R^2 +$$

$$+\frac{m^4}{4} - \frac{1}{2}\left(\frac{1}{6} - \xi\right) m^2 R + \frac{1}{12}\left(\frac{1}{6} - \xi\right)\Box R.$$

Remaining freedom

In the trace $c\Box R$. Wald's fifth axiom does not hold!

- We can add conserved tensors t_{ab} build out of curvature only.
- It must behave as T_{ab} under *"scale"* transformations.
- Some possibilities arises from the variation of

$$t_{ab} = \frac{\delta}{\delta g^{ab}} C \int \sqrt{g} R^2 + D \int \sqrt{g} R_{ab} R^{ab}$$

- ${t_a}^a = \alpha \Box R$
- We use this freedom to cancel the $\Box R$ term from $\langle T \rangle$.

Wald's fifth axiom partially holds for $\langle T'_{ab} \rangle = \langle T_{ab} \rangle - c t_{ab}$

$f(R)$ gravity .....
NB: t_{ab} alone does not guaranty stable solutions.
[Cognola Elizalde Odintsov Zerbini 05, Cognola Zerbini 06]

Equation of the universe

Assuming $\kappa = 0$, we write the equation $-R = 8\pi\langle T\rangle$ as follows

$$-6\left(\dot{H} + 2H^2\right) = 8\pi G\left(-3\left(\frac{1}{6} - \xi\right)\Box - m^2\right)\langle\phi^2\rangle_\omega +$$

$$+\frac{G}{\pi}\left(-\frac{1}{30}\left(\dot{H}H^2 + H^4\right) + 9\left(\frac{1}{6} - \xi\right)^2\left(\dot{H}^2 + 4H^2\dot{H} + 4H^4\right)\right)$$

$$+\frac{G}{\pi}\left(\frac{m^4}{4} - 3\left(\frac{1}{6} - \xi\right)m^2\left(\dot{H} + 2H^2\right)\right)$$

If $\xi = 1/6$, namely for the conformal coupling it simplifies a lot:

$$-6\left(\dot{H} + 2H^2\right) = -8\pi G m^2\langle\phi^2\rangle_\omega + \frac{G}{\pi}\left(-\frac{1}{30}\left(\dot{H}H^2 + H^4\right) + \frac{m^4}{4}\right)$$

Conformal invariant theory

If $\xi = \frac{1}{6}$, $m^2 = 0$, the equation does not depend on the state.

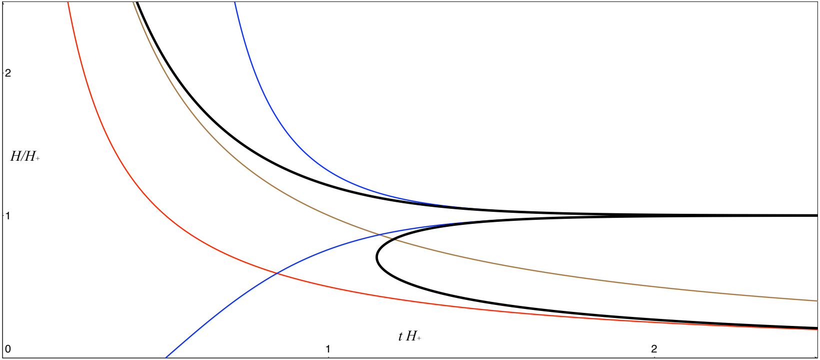
$$\dot{H}\left(H^2 - \frac{H_c^2}{2}\right) = -H^4 + H_c^2 H^2, \qquad H_c^2 = \frac{360\pi}{G}$$

$H^2 = H_c^2$ and $H^2 = 0$ are solutions *(de Sitter, Minkowksi)*.
They are both stable as seen by the full solution

$$Ce^{4t} = e^{2/H}\left|\frac{H + H_c}{H - H_c}\right|^{1/H_c}$$

- It is as in the Starobinsky model but now with stable de Sitter. *[Starobinsky 80, Vilenkin 85]*

Clearly $H = 0$ and $H = H_c = H_+$ are stable solutions.



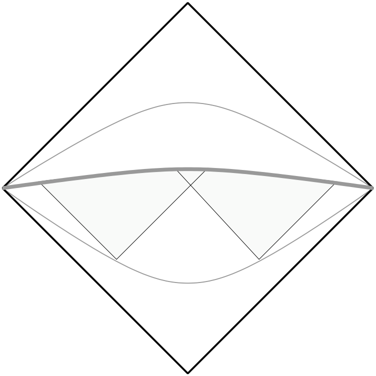
- $H = H_c$ is order of magnitude to big to describe the present expansion velocity of the universe.
- Two fixed points instead of one, a length scale is introduced (proportional to G).

Particle horizon

$\mathbb{R}^+ \times \Sigma$, in t_0 singularity. $\tau = \int_t^{t_1} \frac{dt}{a(t)}$ Where is $\tau(t_0)$?

$ds^2 = -dt^2 + a^2 d\mathbf{x}^2$ $ds^2 = a^2 \left(-d\tau^2 + d\mathbf{x}^2\right)$

Maximal **comoving distance** (if $c = 1$) it is τ.



- Radiation dominated: $\tau = \tau_0 - A(t - t_0)^{1/2} \to \tau_0$ for $t \to t_0$
- Matter dominated: $\tau = \tau_0 - A(t - t_0)^{1/3} \to \tau_0$ for $t \to t_0$
- $\rho = 1/a(t)^2$ $\tau = \tau_0 - \log(t - t_0) \to -\infty$ for $t \to t_0$

Massive model

Important: The quantum states enter in the equation.
$\langle\phi^2\rangle = \frac{[w]}{8\pi^2} + Am^2 + BR.$ We assume $[w] = 0$.

$$\dot{H}\left(H^2 - H_0^2\right) = -H^4 + 2H_0^2 H^2 + M$$

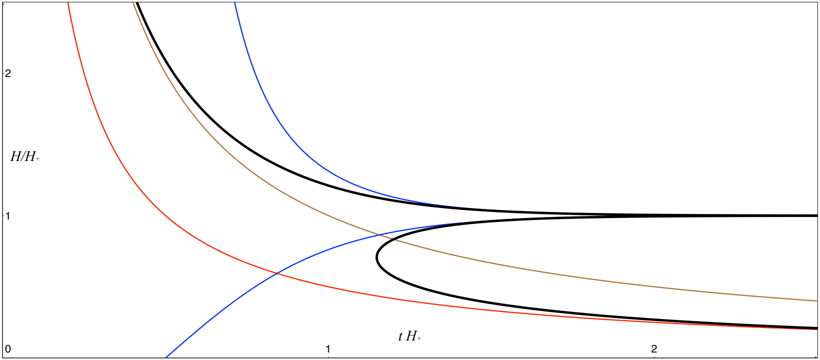
where H_0 and M are two constants with the following values

$$H_0^2 = \frac{180\pi}{G} - B, \qquad M = \frac{15}{2}m^4 - 240\pi^2 m^4 A$$

At most two fixed point (de Sitter phases)

$$H_\pm^2 = H_0^2 \pm \sqrt{H_0^4 + M},$$

they appear to be both stable.
We want to have Minkowski $H_- = 0, \Longrightarrow A = (32\pi^2)^{-1}$.
Freedom in m and B to *"Fine tune"* H_+.



- H_+ can be made small by suitable choices of m^2 and A, B
- It could model dark energy.
- Quantum effects are hardly negligible.
- Smooth exit form rapid expansion in the past. *[Shapiro Sola 02]*

Massive models and adiabatic vacuum $\langle \phi^2 \rangle$

On the Bunch-Davies state in dS $\langle \phi^2 \rangle$ is a constant.

We would like to choose **“ground states”**.

Impossible. Adiabatic states, have similar properties.
[Parker, Parker and Fulling, Lüders Roberts, Junker Schrohe, Olbermann]

- Minimize the particle creation rate. *[Parker]*
- Are states of minimal energy in the sense of Fewster. *[Olbermann]*
- They can be thought as approximated ground states.

Let’s see how they are constructed.

Two-points function

We consider $\xi = 1/6$, we use the conformal time:

$$\tau = \int \frac{dt}{a(t)}, \qquad f'(\tau) = a(\tau)\dot{f}(\tau)$$

Two-points function:

$$\omega(x_1, x_2) = \frac{1}{8\pi^3} \frac{1}{a(\tau_1)a(\tau_2)} \int d^3\mathbf{k}\, \overline{T_k(\tau_1)}\, T_k(\tau_2) e^{i\mathbf{k}\cdot(\mathbf{x}_1 - \mathbf{x}_2)};$$

with

$$T_k'' + k^2 T_k + m^2 a(\tau)^2 T_k = 0, \qquad \overline{T_k}\, T_k' - T_k \overline{T_k}' = i.$$

WKB approximation:

$$T_k(\tau) = \frac{1}{\sqrt{2\Omega_k(\tau)}} e^{i\int_{\tau_0}^{\tau} \Omega_k(\tau')d\tau'}.$$

Recursive relations

Then Ω_k^2 must satisfy the following equation

$$\Omega_k^2 = k^2 + m^2 a(\tau)^2 + \frac{5}{16}\left(\frac{(\Omega_k^2)'}{\Omega_k^2}\right)^2 - \frac{1}{4}\frac{(\Omega_k^2)''}{\Omega_k^2}$$

Recursively ${\Omega_k^{(0)}}^2 = k^2 + m^2 a(\tau)^2$ and $\Omega_k^{(n+1)}$ plugging $\Omega_k^{(n)}$ on the right.
The approx. two-point function is $\omega_2^{(n)}(x,y)$ is found using $\Omega_k^{(n)}$.

$$\langle \phi^2 \rangle_{(n)}(x) = \lim_{y\to x}(\omega_2^{(n)}(x,y) - H(x,y)) + \alpha'' R + \beta'' m^2$$

$\omega^{(n)}$ becomes closer to an Hadamard state $n \to \infty$.

The expectation value of $\langle\phi^2\rangle$

If $\Omega^{(n)^2} \geq 0$

$$\langle\phi^2\rangle_{(n)} = \frac{1}{4\pi^2\ a(\tau)^2}\int_0^\infty dk\ k^2\left(\frac{1}{\Omega_k^{(n)}(\tau)} - \frac{1}{\Omega_k^{(0)}(\tau)}\right) + \alpha' R + \beta' m^2.$$

Problem: we have a good control only in the $k >> 1$.
We expand the integral in powers of $1/m^2$

$$\langle\phi^2\rangle_{(n)} = \alpha m^2 + \beta R + O\left(\frac{1}{m^2}\right)$$

The regime $m^2 >> R$ is what we need. If $m = 1GeV$ $\frac{m^2}{R} \sim 10^{82}$

Summary

- Semiclassical solutions of Einstein's equation.
- The solutions depend upon the quantum states.
- The de Sitter phases could be stable only fixing the renormalization freedom.
- That solution shows a phase of rapid expansion at the beginning.

Open Questions

- How can we choose a nicer state?
- Fluctuations?
- Connection with $f(R)$ gravity?
- Origin of R^2 terms in the action? An hint on quantum gravity?