

Gauge theories in
local causal perturbation theory

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Franz-Marc Boas
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Abstract

In this thesis quantum gauge theories are considered in the framework of local, causal perturbation theory. Gauge invariance is described in terms of the BRS formalism. Local interacting field operators are constructed perturbatively and field equations are established. A nilpotent BRS transformation is defined on the local algebra of fields. It allows the definition of the algebra of local observables as an operator cohomology. This algebra of local observables can be represented in a Hilbert space.

The interacting field operators are defined in terms of time ordered products of free field operators. For the results above to hold the time ordered products must satisfy certain normalization conditions. To formulate these conditions also for field operators that contain a spacetime derivative a suitable mathematical description of time ordered products is developed.

Among the normalization conditions are Ward identities for the ghost current and the BRS current. The latter are generalizations of a normalization condition that is postulated by Dütsch, Hurth, Krahe and Scharf for Yang-Mills theory. It is not yet proven that this condition has a solution in every order. All other normalization conditions can be accomplished simultaneously.

A principle for the correspondence between interacting quantum fields and interacting classical fields is established. Quantum electrodynamics and Yang-Mills theory are examined and the results are compared with the literature.

Zusammenfassung

In dieser Arbeit werden Quanten-Eichtheorien im Rahmen der lokalen, kausalen Störungstheorie behandelt. Eichinvarianz wird mit Hilfe des BRS-Formalismus beschrieben. Lokale, wechselwirkende Feldoperatoren werden störungstheoretisch konstruiert und Feldgleichungen zwischen ihnen werden hergeleitet. Eine nilpotente BRS-Transformation wird auf der lokalen Feld-Algebra definiert. Sie gestattet die Definition der lokalen Observablen-Algebra als eine Operator-Kohomologie. Diese lokale Observablen-Algebra besitzt eine Hilbertraum-Darstellung.

Die wechselwirkenden Feldoperatoren werden mit Hilfe zeitgeordneter Produkte freier Feldoperatoren definiert. Damit die obigen Resultate gelten, müssen die zeitgeordneten Produkte bestimmte Normierungsbedingungen erfüllen. Um diese Bedingungen auch für Felder mit Raum-Zeit-Ableitungen formulieren zu können, wird eine geeignete mathematische Beschreibung zeitgeordneter Produkte entwickelt.

Unter den Normierungsbedingungen sind Ward-Identitäten für den Geist-Strom und den BRS-Strom. Letztere sind Verallgemeinerungen einer Normierungsbedingung, die Dütsch, Hurth, Krahe und Scharf für die Yang-Mills-Theorie fordern. Es ist noch nicht bewiesen, daß diese Bedingung in jeder Ordnung eine Lösung besitzt. Alle anderen Normierungsbedingungen können gleichzeitig erfüllt werden.

Ein Prinzip für die Korrespondenz zwischen wechselwirkenden Quantenfeldern und wechselwirkenden klassischen Feldern wird aufgestellt. Quanten-Elektrodynamik und Yang-Mills-Theorie werden untersucht, und die Ergebnisse werden mit der Literatur verglichen.

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1. INTRODUCTION

Four fundamental interactions in nature are known today: Gravitation, electrodynamics, weak and strong nuclear forces. The latter three are in present day elementary particle physics successfully described by quantum gauge field theories. Successfully means in this context that there are no experimental data that do not agree with the predictions of these theories and that the agreement is very good e.g. in quantum electrodynamics (QED). The general theory of relativity describes gravitation classically. It is also a gauge theory in a wider sense of the word. A sound quantum theory for gravitation is still missing.

The distinguishing feature of these gauge theories is their gauge group: $SU(2)\times U(1)$ for the combined theory of electric and weak interactions and $SU(3)$ for the strong interaction. Both gauge groups are non-Abelian Lie groups. Therefore a comprehensive understanding of non-Abelian quantum gauge theories is needed to understand nature at the quantum level.

Originally the conceptual and mathematical framework of quantum field theory was developed for Abelian theories and in particular for QED. This was already an established theory in perfect agreement with the experimental data when physicists directed their attention towards non-Abelian gauge theories. They realized that quantum field theory required a modification of its mathematical description before it could be applied to non-Abelian theories.

The first study of a non-Abelian model — motivated by the isospin $SU(2)$ group — which attained wide reception was done by Yang and Mills [YM54] in 1954¹. The interest of elementary particle physicists in non-Abelian quantum field theories grew strongly when in the next two decades several other such models were proposed to explain various phenomena. These include e.g. the Salam–Weinberg model [Sal68, Wei67] and the $SU(3)$ colour model for the strong interaction [GM64, Zwe64], but also attempts to quantize gravitation, e.g. [Fey63] or [DeW67a, DeW67b, DeW67c]. There was a series of obstacles to a satisfactory quantum theory for non-Abelian gauge theories due to the self coupling of the gauge bosons. A naive application of the methods developed for QED leads to serious difficulties, like an S-matrix that fails to be unitary [Fey63, DeW67b].

A major step to overcome these obstacles was made by Faddeev and Popov [FP67, Fad69]. They defined a unitary S-matrix in the functional integral approach, but for that they had to introduce unphysical fields that violate the spin-statistics theorem — the famous Faddeev Popov ghosts. In the mid seventies Becchi, Rouet and Stora [BRS74, BRS76] and independently of them Tyutin [Tyu75] found that the Faddeev Popov Lagrangian is invariant under a rigid symmetry transformation that mixes the ghosts with the other fields — the BRS transformation. Kugo and Ojima [KO79] gave an operator formulation² for this BRS theory, and Scharf and collaborators [DHKS94a] found with the operator gauge invariance a criterion of BRS symmetry for operator theories that needs no recurrence to an underlying classical theory³.

¹The first who studied non-Abelian models was O. Klein in 1938 [Kle39]

²Curci and Ferrari [CF76] gave already an operator formulation, but they postulated wrong hermiticity properties for the ghosts

³Originally operator gauge invariance was postulated for theories of the Yang-Mills type. Recent results of Scharf and Wellmann [SW99] that it also a suitable criterion for spin two models

Quantum field theory is plagued with two sources of possible infinities: the ultraviolet and the infrared divergences. Ultraviolet divergences are due to the distributional character of the field operators. In perturbation theory they can be removed by numerous renormalization procedures — so they are under control in this framework. Unfortunately these renormalization procedures are not unique — there remains the freedom of finite renormalization.

Infrared divergences occur since the asymptotic behaviour of incoming and outgoing interacting fields is not under control. This problem is particularly severe for non-Abelian gauge theories. It may be overcome by a replacement of the coupling constant by a spacetime dependent switching function so that the theory becomes free at finite times in the past and in the future. But as the real physical coupling is constant, one must in general perform the adiabatic limit, i.e. let the switching function tend to a constant. This limit does not exist in general.

In QED the infrared divergences are logarithmic, and Blanchard and Seneor [BS75] proved that the adiabatic limit exists for Green's and Wightman functions. Unfortunately this is no longer true for non-Abelian theories. Their infrared behaviour is in general worse.

For strongly interacting fields this comes from the experimental observation of confinement. This means the fact that strongly interacting particles always combine to hadrons. Even after a high energy scattering process that breaks up the hadron structure the particles recombine immediately into new hadrons (hadronization). So the fields are not asymptotically free but constitute bound states. Moreover confinement cannot be described perturbatively.

In the electroweak theory confinement does not occur, but the model contains unstable, observable particles — the vector bosons W^\pm and Z . These cannot occur as asymptotic states.

A solution for the infrared problem is to consider local theories, i.e. theories where all fields are localized in a finite region of spacetime. If the coupling is constant within this region and if the algebra of fields remains unaltered when the coupling is modified outside that region, the adiabatic limit needs not to be performed. Brunetti and Fredenhagen [BF97] proved that such a modification induces merely a unitary transformation on the algebra of fields. So the physical content is not changed by that modification and there is consequently no need for the adiabatic limit. Therefore no infrared problems occur in the construction of the local algebras.

One common problem of gauge theories — already encountered in QED — is that the algebra of fields must be quantized in an indefinite inner product space. Therefore positivity must be assured, i.e. the algebra of observables must be non trivially represented in a Hilbert space. Dütsch and Fredenhagen [DF99] succeeded in proving positivity for perturbation theories quantized in the BRS framework, provided the underlying free theory is also positive. In their view the interacting theory is regarded as a deformation of that underlying free model. They also constructed a local perturbation theory for QED.

The first to examine Yang-Mills theories in the causal framework were Scharf and collaborators [DHKS94a] - [DHS95b], see also [Sch95]. They investigated the operator gauge invariance in the Yang-Mills case and found that it can be accomplished, provided a weak assumption concerning the infrared behaviour of the Green's functions is fulfilled.

The aim of this thesis is to construct local perturbative gauge theories as operator

theories in the BRS framework. The design is as general as possible, the motivation is Yang-Mills theory which serves as an example throughout the thesis.

The result is that the construction can always be performed, provided the generalized operator gauge invariance holds. It could not be proven that the latter can be accomplished in general models. A similar set of equations are the *descent equations* in the framework of algebraic renormalization — see, e.g. [PS95]. It may be possible to prove generalized operator gauge invariance by translating these results into our language, but this seems to be a tedious task and is not done here.

We use the renormalization scheme of causal perturbation theory as it was developed by Epstein and Glaser [EG73] following ideas proposed by Bogoliubov, Shirkov [BS59] and Stückelberg. It avoids divergent expressions throughout the entire procedure. Scharf and collaborators as well as Dütsch and Fredenhagen formulated their results in the same framework. This makes it easy to use their results for our construction and to compare them with our results.

Moreover our approach is local in order to avoid infrared divergences and to be able to define observables and physical states.

Like Dütsch and Fredenhagen we use normalization conditions for the time ordered products as an essential tool to establish desired relations in the field algebra. Their normalization conditions are generalized to include fields that contain a spacetime derivative. Ward identities for the ghost and BRS current are introduced as new normalization conditions with regard to the definition of observables and physical states. We introduce an algebra of auxiliary variables for the fields containing a spacetime derivative and define a linear representation of the polynomials in this algebra as operators acting on the Fock space. We present a reformulation of time ordering. It is formally a multi linear generalization of the linear representation mentioned above to multiple arguments. This and the definition of propagator functions for the fields with a spacetime derivative allows us to generalize the normalization conditions in the desired manner. It is proven that all these conditions — except the BRS Ward identities — can be accomplished simultaneously. The existence of a solution for the BRS Ward identities and its compatibility with the other conditions must be proven in individual models. The proof for QED is presented.

There are relations for the local field algebra that are determined by the normalization conditions, e.g. renormalized field equations and the BRS algebra. The latter allows for a definition of observables and a construction of a positive physical state space.

The thesis is organized as follows: In chapter (2) we set up the algebraic framework of BRS theory, following Kugo and Ojima [KO79]. The definition of observables and the construction of the Hilbert space are performed using certain algebraic relations between the interacting operators. The rest of the thesis will be devoted to the construction of models in which these relations hold.

In chapter (3) the free model underlying our perturbation theory is put up. The algebra of auxiliary variables is constructed and its linear representation as Fock space operators is defined. Then the propagator functions are examined. Finally the proof of Razumov and Rybkin [RR90] for the positivity of theories with certain BRS charges is presented.

The new definition of time ordering is given in chapter (4). It contains also the formulation of six normalization conditions and the proof that the first five have

simultaneous solutions. The sixth, the BRS Ward identities, is shown to be equivalent with a generalized version of operator gauge invariance.

Local causal perturbation theory is introduced in chapter (5) along the lines of Epstein, Glaser [EG73], Dütsch and Fredenhagen [DF99]. Conditions for a polynomial to be a candidate for a Lagrangian are given.

The local field algebra is constructed in chapter (6). The conserved currents and charges, the ghost number of an interacting field and the interacting BRS transformation are defined, field equations and the BRS algebra are derived. The chapter concludes with a reflection on the correspondence between the quantum theory defined above and its classical counterpart.

The inspection of gauge theories is deepened in chapter (7) for two exemplary models: QED and Yang-Mills theory. The BRS Ward identities are proven for QED, and we compare the relations between the interacting fields with those between the corresponding classical fields.

At the end a conclusion and an outlook for possible further developments are included.

2. BRS THEORY — ALGEBRAIC CONSIDERATIONS

In this chapter canonical BRS theory according to Kugo and Ojima [KO79] is carried through on a purely algebraic level. The availability of suitable BRS and ghost charges is formulated as assumptions. Then perturbative theories — i.e. theories where the operators and the state vectors are formal power series — are examined in this framework. Dütsch and Fredenhagen [DF99, DF98] prove that the positivity structure of a theory can be maintained during deformation. Their proof is presented here.

2.1. Why BRS theory? All quantum gauge theories share one common difficulty: There is no positive definite Hilbert space in which the field algebra can be represented and which possesses a nontrivial unitary representation of the Poincaré group. Nakanishi and Ojima [NO90] proved that there exists no nontrivial Hilbert space representation for manifestly covariant theories with massless gauge bosons. This could be circumvented by non covariant gauges, but this means abandoning manifest covariance.

The field algebra is not observable, so a direct physical interpretation of the theory which requires a Hilbert space representation is not possible. But the algebra of observables must have a Hilbert space representation, and the Hilbert space must carry a unitary representation of the Poincaré group.

For QED Gupta [Gup50] and Bleuler [Ble50] found an elegant way out of this dilemma. They retain manifest covariance at the prize of representing the field algebra in an indefinite inner product space. Then there exists a non trivial, pseudo unitary⁴ representation of the Poincaré group. This space is too big: It contains vectors with negative norm that have no physical interpretation — they would lead to negative transition probabilities. Consequently the physical state vectors form a distinguished proper subspace of the inner product space. This subspace is selected by a linear subsidiary condition, and it is found to be positive semidefinite. It becomes a Hilbert space with unitary action of the Poincaré group when all state vectors differing by a zero norm vector are identified with each other and the space is subsequently completed.

Unfortunately this strategy breaks down in non Abelian gauge theories because there is no appropriate subsidiary condition available. This is due to the nonlinear self interaction of the gauge fields.

BRS theory is a solution for that problem. The canonical BRS formalism of Kugo and Ojima [KO79] follows the same ideas as Gupta and Bleuler but it can also be applied to non-Abelian theories. Initially the algebra of fields is again represented in an indefinite inner product space. The presence of the ghosts in the BRS approach makes it possible to define a suitable subsidiary condition for the physical subspace which is a Hilbert space. The formalism provides also a definition of an algebra of observables that is represented in this Hilbert space. There exists a pseudo unitary action of the Poincaré group on the indefinite space. This action is lifted to a unitary one on the Hilbert space.

2.2. Canonical BRS theory. The construction starts in the following situation: There is an initial Hilbert space $\{\mathcal{V}, (\cdot, \cdot)\}$ with a positive scalar product (\cdot, \cdot) that encompasses all fields including the unphysical ones (scalar vector bosons, ghosts

⁴Pseudo unitary, pseudo hermitian etc. means unitary, hermitian etc. w.r.t. the indefinite inner product.

etc.). This scalar product has no direct physical meaning. It does not describe the transition amplitudes, in particular it is not Poincaré covariant. The adjoint in this Hilbert space is denoted as $^+$, i.e. $(\phi, A\psi) = (A^+\phi, \psi)$ for every⁵ $A \in \text{End } \mathcal{V}$. It is possible to find a Krein operator $J \in \text{End}(\mathcal{V})$ in the Hilbert space with the following three properties:

- J is hermitian, i.e. $J^+ = J$
- It is idempotent, i.e. $J^2 = \mathbb{1}$
- It defines a new inner product on \mathcal{V} via

$$\langle \phi, \psi \rangle \stackrel{\text{def}}{=} (\phi, J\psi) \quad (2.1)$$

such that the new inner product is Poincaré covariant.

The new inner product is assumed to describe the correct transition probabilities. Therefore it is referred to as the physical inner product. The vector space \mathcal{V} forms a Krein space with the physical product $\langle \cdot, \cdot \rangle$. Since (\cdot, \cdot) was not covariant while $\langle \cdot, \cdot \rangle$ was, $J = \mathbb{1}$ can be excluded. Then the physical inner product is always indefinite, because there must exist a vector $|\phi\rangle$ such that $(\mathbb{1} - J)|\phi\rangle \neq 0$, and then $(\mathbb{1} - J)|\phi\rangle$ has negative norm. The adjoint w.r.t. the physical inner product is defined as an involution denoted by $*$, namely $A^* \stackrel{\text{def}}{=} JA^+J$, such that $\langle \phi, A\psi \rangle = \langle A^*\phi, \psi \rangle$ for every $A \in \text{End } \mathcal{V}$.

For the canonical BRS theory the following assumption is essential:

A1: There exists an operator $Q_B \in \text{End}(\mathcal{V})$ — the BRS charge — with the following properties:

- Q_B is a conserved charge.
- It is pseudo hermitian, i.e. $(Q_B)^* = Q_B$.
- It is nilpotent⁶, i.e. $(Q_B)^2 = 0$.
- It annihilates the vacuum, i.e. $Q_B|\omega\rangle = 0$

where $|\omega\rangle$ is the vacuum vector. This assumption is highly non trivial, and the appearance of ghosts in \mathcal{V} is necessary for it. It has to be verified in the concrete model.

It is easily verified that the image of Q_B contains only zero norm vectors w.r.t. the physical scalar product:

$$\langle Q_B\phi, Q_B\phi \rangle = \langle \phi, (Q_B)^2\phi \rangle = 0. \quad (2.2)$$

With the second assumption a grading is introduced on \mathcal{V} by means of the ghost charge Q_c .

A2: There exists an operator $Q_c \in \text{End}(\mathcal{V})$ — the ghost charge — with the following properties:

- Q_c is a conserved charge.
- It is anti pseudo hermitian, i.e. $(Q_c)^* = -Q_c$.
- It has integer eigenvalues, i.e. $Q_c|\psi\rangle = q|\psi\rangle \implies q \in \mathbb{Z}$.
- It satisfies the commutator relation $[Q_c, Q_B]_- = Q_B$.
- It annihilates the vacuum, i.e. $Q_c|\omega\rangle = 0$.

The eigenvalue of a state vector w.r.t. the ghost charge is called its ghost number. For the physical inner product of two vectors to be non zero they must have opposite

⁵ $\text{End}(\mathcal{V})$ is the space of endomorphisms on \mathcal{V}

⁶Nilpotent means throughout this thesis nilpotent of order two.

ghost numbers: Let $Q_c \psi = q \psi$ and $Q_c \phi = p \phi$, then

$$0 = \langle \psi, Q_c \phi \rangle - \langle \psi, Q_c \phi \rangle = \langle \psi, Q_c \phi \rangle + \langle Q_c \psi, \phi \rangle = (q + p) \langle \psi, \phi \rangle, \quad (2.3)$$

so $(q + p) = 0$ or $\langle \psi, \phi \rangle = 0$. This implies in particular that only states with vanishing ghost number can have non zero norm w.r.t. the physical inner product.

The commutator relation $[Q_c, Q_B]_- = Q_B$ forms together with the nilpotency of the BRS charge, $(Q_B)^2 = 0$, the *BRS algebra*.

Like in the Gupta-Bleuler scheme the negative norm states are excluded by a subsidiary condition. The kernel of Q_B is regarded as a candidate for the physical Hilbert space. It contains necessarily zero norm states from the image of Q_B — due to $(Q_B)^2 = 0$ we have $\text{im } Q_B \subset \ker Q_B$ — and possibly also vectors with non vanishing ghost number. Therefore the following definition for the Hilbert space \mathcal{H}_{ph} of physical state vectors is given:

$$\mathcal{H}_{\text{ph}} \stackrel{\text{def}}{=} \overline{(\ker Q_B, \mathcal{V}) / (\text{im } Q_B, \mathcal{V})}^{\|\cdot\|}. \quad (2.4)$$

Completion is understood in the norm topology. Now it must be verified that the physical state vectors form a positive definite inner product space. This is guaranteed if the following positivity assumption is valid.

A3:

- The kernel of Q_B contains only positive semidefinite vectors, i.e. $Q_B |\phi\rangle = 0 \implies \langle \phi, \phi \rangle \geq 0$
- Its image encompasses all zero norm vectors in its kernel, i.e. $|\phi\rangle \in \ker(Q_B, \mathcal{V})$ and $\langle \phi, \phi \rangle = 0 \implies |\phi\rangle \in (\text{im } Q_B, \mathcal{V})$.

The second point guarantees in particular that all elements in $(\ker Q_B, \mathcal{V})$ with nonvanishing ghost number are in $(\text{im } Q_B, \mathcal{V})$. The scalar product is well defined on these equivalence classes, so it does not depend on the representative of a class:

$$\langle \phi + Q_B \chi, \psi \rangle = \langle \phi, \psi \rangle + \langle \chi, Q_B \psi \rangle = \langle \phi, \psi \rangle. \quad (2.5)$$

It is also positive definite by construction — if assumption **A3** holds —, so the quotient space is a pre Hilbert space and becomes a Hilbert space after completion. The structure above is called a state cohomology.

The ghost charge induces a derivation on $\text{End}(\mathcal{V})$,

$$s_c(A) \stackrel{\text{def}}{=} [Q_c, A]_- \quad \forall A \in \text{End}(\mathcal{V}). \quad (2.6)$$

Its eigenvalue for an operator $A \in \text{End}(\mathcal{V})$ is called the ghost number of A and is always an integer.

The BRS charge induces an graded derivation on $\text{End}(\mathcal{V})$, namely the BRS transformation⁷

$$s(A) \stackrel{\text{def}}{=} [Q_B, A]_{\mp} \quad \forall A \in \text{End}(\mathcal{V}). \quad (2.7)$$

It is nilpotent because Q_B is also nilpotent and the Jacobi-identity holds for the graded commutators.

With these definitions the algebra of observables \mathcal{A}_{ph} can be defined as

$$\mathcal{A}_{\text{ph}} \stackrel{\text{def}}{=} \left((\ker s, \text{End}(\mathcal{V})) \cap (\ker s_c, \text{End}(\mathcal{V})) \right) / \left((\text{im } s, \text{End}(\mathcal{V})) \cap (\ker s_c, \text{End}(\mathcal{V})) \right). \quad (2.8)$$

⁷Here $[\cdot, \cdot]_{\mp}$ denotes the graded commutator. Suppose, $A, B \in \text{End } \mathcal{V}$ have ghost numbers $a, b \in \mathbf{Z}$. Then $[A, B]_{\mp} \stackrel{\text{def}}{=} AB - (-1)^{ab} BA$.

This structure is called an operator cohomology. Its elements are well defined operators on \mathcal{H}_{ph} , i.e. $\mathcal{A}_{\text{ph}}(\ker Q_B, \mathcal{V}) \subset (\ker Q_B, \mathcal{V})$ and $\mathcal{A}_{\text{ph}}[0] = [0]$, where $[0]$ is the equivalence class of zero.

There is a *-involution induced on the algebra of observables by the *-involution on the representatives. But unlike the original involution this one acts on operators on a Hilbert space, so the notions hermitian, unitary and so on must be used without the prefix pseudo.

There is also a unitary action of the Poincaré group defined on \mathcal{H}_{ph} , namely the lift of the initial pseudo-unitary action on the representatives to the equivalence classes. This induces a unitary representation on \mathcal{H}_{ph} .

There is a physical interpretation available for the cohomologies. Initially the model is not characterized in terms of the algebra of field operators described here but in terms of the sub algebra without the ghosts — these were only introduced to make possible the definition of the BRS charge. In the picture above physics is invariant under local gauge transformations, i.e. gauge transformations generated by spacetime dependent functions. Then the BRS transformation, restricted to the sub algebra, may be regarded as the infinitesimal local gauge transformation. The role of the spacetime dependent functions is played by the ghosts. For them the BRS transformation is defined such that it is nilpotent on the entire algebra.

So the restriction to the kernel of s singles out fields that are invariant under infinitesimal gauge transformations. Fields in the same equivalence class are regarded as physically indistinguishable. In this interpretation the physical Hilbert space contains equivalence classes of states that are invariant under infinitesimal gauge transformations.

2.3. Interacting theories and deformation stability. In perturbation theory field operators are represented by formal power series of linear operators. This makes it necessary to recapitulate the BRS formalism for formal power series of state vectors and operators, since e.g. the notion of positivity is not defined a priori for formal power series. This situation has been examined by Dütsch and Fredenhagen [DF98, DF99] and we present their results here.

In their picture the interacting theory is a deformation of an underlying free theory. In some models positivity — i.e. assumption **A3** — can be proven by direct computation for the underlying free theory. Dütsch and Fredenhagen found a construction for the deformed — i.e. interacting — state space such that positivity holds also there in a sense defined below.

In the interacting theory both the state space and the operators acting on it are modules over the ring $\tilde{\mathbb{C}}$ of formal power series of complex numbers:

$$\tilde{\mathbb{C}} \stackrel{\text{def}}{=} \left\{ \tilde{a} = \sum_{n=0}^{\infty} g^n a_n : a_n \in \mathbb{C} \right\} \quad (2.9)$$

where g is the deformation parameter. The element $\tilde{\mathbb{1}} \stackrel{\text{def}}{=} (1, 0, 0, \dots)$ is the identity in this ring. An element $\tilde{a} \in \tilde{\mathbb{C}}$ is only invertible⁸ if $a_0 \neq 0$. The interacting indefinite inner product space is defined as the $\tilde{\mathbb{C}}$ -module $\tilde{\mathcal{V}} \stackrel{\text{def}}{=} \left\{ \tilde{\psi} = \sum_n g^n \psi_n : \psi_n \in \mathcal{V} \right\}$ which has the inner product $\langle \cdot, \cdot \rangle$ induced from \mathcal{V} . For $\tilde{\psi} = \sum_n g^n \psi_n$ and $\tilde{\chi} =$

⁸Bordemann and Waldmann [BW96] consider Laurent series instead. These are invertible if $\tilde{a} \neq 0$, so they form a field.

$\sum_n g^n \chi_n$ this means

$$\begin{aligned} \langle \cdot, \cdot \rangle : \tilde{\mathcal{V}} \times \tilde{\mathcal{V}} &\rightarrow \tilde{\mathbb{C}} \\ \langle \tilde{\psi}, \tilde{\chi} \rangle &= \sum_n g^n \left(\sum_{k=1}^n \langle \psi_k, \chi_{n-k} \rangle \right). \end{aligned} \quad (2.10)$$

This is sesquilinear in $\tilde{\mathbb{C}}$, i.e. $\langle \tilde{a}\tilde{\chi}, \tilde{b}\tilde{\psi} \rangle = \tilde{a}^*\tilde{b} \langle \tilde{\chi}, \tilde{\psi} \rangle$. The $*$ means complex conjugation, where the deformation parameter g is real, so

$$\tilde{a}^* = \sum_{n=0}^{\infty} g^n \bar{a}_n \quad (2.11)$$

where $\bar{}$ denotes complex conjugation in \mathbb{C} .

The operators in $\text{End}(\tilde{\mathcal{V}})$ acting on $\tilde{\mathcal{V}}$ can be written as

$$\text{End}(\tilde{\mathcal{V}}) = \left\{ \tilde{A} = \sum_n g^n A_n : A_n \in \text{End}(\mathcal{V}) \right\} \quad (2.12)$$

and form a $\tilde{\mathbb{C}}$ -module, too. The multiplication law in this algebra is

$$\tilde{A} \cdot \tilde{B} = \sum_n g^n \left(\sum_{k=1}^n A_k \cdot B_{n-k} \right) \quad \tilde{A}, \tilde{B} \in \text{End}(\tilde{\mathcal{V}}). \quad (2.13)$$

The interacting BRS-charge and the interacting ghost charge are such operators,

$$\begin{aligned} \tilde{Q}_B &= \sum_n g^n Q_{B,n}, \quad Q_{B,n} \in \text{End}(\mathcal{V}) \\ \text{and} \quad \tilde{Q}_c &= \sum_n g^n Q_{c,n}, \quad Q_{c,n} \in \text{End}(\mathcal{V}), \end{aligned} \quad (2.14)$$

where $\tilde{Q}_{B,0}$ and $\tilde{Q}_{c,0}$ agree with the free charges. \tilde{Q}_B must be chosen such that it is nilpotent, $\tilde{Q}_B^2 = 0$, and pseudo hermitian, $(\tilde{Q}_B)^* = \tilde{Q}_B$, and \tilde{Q}_c must be anti pseudo hermitian, $(\tilde{Q}_c)^* = -\tilde{Q}_c$. The involution $*$ is the one induced from $\text{End}(\mathcal{V})$.

The charges must satisfy the BRS algebra $[\tilde{Q}_c, \tilde{Q}_B]_- = \tilde{Q}_B$.

The interacting state space can be defined as in the general case,

$$\tilde{\mathcal{H}}_{\text{ph}} \stackrel{\text{def}}{=} (\ker \tilde{Q}_B, \tilde{\mathcal{V}}) / (\text{im } \tilde{Q}_B, \tilde{\mathcal{V}}), \quad (2.15)$$

with the only difference that the space is not completed since there is no convenient topology in the space of formal power series.

The question is whether this space has a positive scalar product, and above all what positivity means for formal power series.

Following Dütsch and Fredenhagen [DF99] we adopt here Steinmann's [Ste89] point of view⁹ that a formal power series $\tilde{b} = \sum_n b_n g^n \in \tilde{\mathbb{C}}$ is *positive* if it is the absolute square of another power series $\tilde{c} \in \tilde{\mathbb{C}}$, i.e. $\tilde{b} = \tilde{c}^* \tilde{c}$. Dütsch and Fredenhagen define also that a class of state vectors $[\tilde{\varphi}] \in \tilde{\mathcal{H}}_{\text{ph}}$ can be normalized if there exists an

⁹Here Bordemann and Waldmann [BW96] follow again a different prescription: They define a real formal power series as positive if its first non vanishing coefficient is a positive number. With this definition the field of real Laurant series becomes ordered. The notion of positivity presented here is a stricter one: Every positive series in Steinmann's sense is also positive in their sense, but not converse.

$\tilde{a} \in \tilde{\mathbf{C}}$ and $[\tilde{\psi}] \in \tilde{\mathcal{H}}_{\text{ph}}$ such that $[\tilde{\varphi}] = \tilde{a}[\tilde{\psi}]$ and $\langle [\tilde{\psi}], [\tilde{\psi}] \rangle = \tilde{\mathbb{1}}$.

With these notions of positivity and normalizability they prove in [DF99] the following results:

Let the positivity assumption **A3** be fulfilled for the undeformed theory. Then

$$(i) \quad \langle \tilde{\psi}, \tilde{\psi} \rangle \geq 0 \quad \forall \tilde{\psi} \in (\ker \tilde{Q}_B, \tilde{\mathcal{V}}) \quad (2.16)$$

$$(ii) \quad \tilde{\psi} \in (\ker \tilde{Q}_B, \tilde{\mathcal{V}}) \wedge \langle \tilde{\psi}, \tilde{\psi} \rangle = 0 \implies \tilde{\psi} \in (\text{im } \tilde{Q}_B, \tilde{\mathcal{V}}), \quad (2.17)$$

$$(iii) \quad \forall \psi \in (\ker Q_B, \mathcal{V}) \quad \exists \tilde{\psi} \in (\ker \tilde{Q}_B, \tilde{\mathcal{V}}) : (\tilde{\psi})_0 = \psi \quad (2.18)$$

$$(iv) \quad \text{Every } [\tilde{\psi}] \neq 0 \in \tilde{\mathcal{H}}_{\text{ph}} \text{ is normalizable in the sense above.} \quad (2.19)$$

For the proofs of these results we refer to their article.

So assumption **A3** is fulfilled for the interacting theory if it is fulfilled for the free theory underlying it. Therefore the interacting physical state space defined above is a pre Hilbert space. Result (iii) implies that an interacting vacuum state $|\tilde{\omega}\rangle$ can be defined that is annihilated by \tilde{Q}_B such that $|\tilde{\omega}\rangle_0 = |\omega\rangle$, provided that the free charge annihilates the free vacuum.

The interacting BRS-transformation is the formal power series

$$\tilde{s} = \sum_n g^n s_n, \quad \tilde{s}(\tilde{A}) \stackrel{\text{def}}{=} [\tilde{Q}_B, \tilde{A}]_{\mp} \quad \forall \tilde{A} \in \text{End } \tilde{\mathcal{V}}. \quad (2.20)$$

Each s_n is an anti-derivation on $\text{End } \tilde{\mathcal{V}}$ and s_0 agrees with the free BRS-transformation. \tilde{s}_c is analogously defined as

$$\tilde{s}_c = \sum_n g^n s_{c,n}, \quad \tilde{s}_c(\tilde{A}) \stackrel{\text{def}}{=} [\tilde{Q}_c, \tilde{A}]_{\mp} \quad \forall \tilde{A} \in \text{End } \tilde{\mathcal{V}} \quad (2.21)$$

where each $s_{c,n}$ is a derivation on $\text{End } \tilde{\mathcal{V}}$ and $s_{c,0}$ agrees with s_c . The interacting observable algebra is defined as

$$\tilde{\mathcal{A}}_{\text{ph}} \stackrel{\text{def}}{=} \left((\ker \tilde{s}, \text{End } \tilde{\mathcal{V}}) \cap (\ker \tilde{s}_c, \text{End } \tilde{\mathcal{V}}) \right) / \left((\text{im } \tilde{s}, \text{End } \tilde{\mathcal{V}}) \cap (\ker \tilde{s}_c, \text{End } \tilde{\mathcal{V}}) \right). \quad (2.22)$$

So in the framework of BRS theory an algebra of interacting observables can be defined and represented in a (pre) Hilbert space if the following conditions can be accomplished:

1. In the underlying free theory a ghost charge Q_c and a BRS charge Q_B can be defined that fulfill the assumptions **A1** - **A3**.
2. A conserved interacting BRS charge \tilde{Q}_B can be constructed such that $(\tilde{Q}_B)_0 = Q_B$ with the properties $\tilde{Q}_B^2 = 0$ and $(\tilde{Q}_B)^* = \tilde{Q}_B$.
3. A conserved interacting ghost charge \tilde{Q}_c with integer eigenvalues can be constructed such that $(\tilde{Q}_c)_0 = Q_c$ with the property $(\tilde{Q}_c)^* = -\tilde{Q}_c$.
4. The BRS algebra $[\tilde{Q}_c, \tilde{Q}_B]_{-} = \tilde{Q}_B$ holds.

3. THE FREE THEORY

We start our considerations concerning BRS theory with free theories. The treatment of free theories in the BRS framework is not a goal in its own but provides us with definitions that will become important for the interacting theory in the next chapters. Furthermore positivity is proven for the underlying free model in order to take advantage of deformation stability for the interacting theory.

We already pointed out the essential significance of normalization conditions for the time ordered products in our construction. For some of these normalization conditions it is necessary to give a precise meaning to expressions like $\frac{\partial A}{\partial \varphi_j}(x)$, the derivative of a Wick monomial A w.r.t. a free field operator φ_j . Dütsch and Fredenhagen [DF99] solve this problem for QED by an implicit definition,

$$[A(x), \varphi_j(y)]_{\mp} = i \sum_k \Delta_{jk}(x-y) \frac{\partial A}{\partial \varphi_k}(x) \quad (3.1)$$

where $\Delta_{jk}(x)$ is a commutator function. This equation is indeed a definition for the partial derivative on the right hand side if the theory contains no derivated fields¹⁰ like QED. But for theories that do contain such derivated fields — like Yang-Mills theory — there is no such definition available.

The natural attempt to include derivated fields would be the replacement of the partial derivative by a functional derivative, where the latter would be defined by means of

$$[A(x), \varphi_j(y)]_{\mp} = i \sum_k \int d^4z \Delta_{jk}(z-y) \frac{\delta A(x)}{\delta \varphi_k(z)}. \quad (3.2)$$

Unfortunately the equation above is no definition. This can be seen as follows: Let D be the differential operator that implements the field equations for the free fields such that

$$\sum_j D_{ij}^x \Delta_{jk}(x) = 0. \quad (3.3)$$

Such an operator exists in general, it will be explicitly constructed later in this chapter. We can define an operator D^* according to

$$\int d^4z f(z) (D_{ij}^{*,z} g(z)) = \int d^4z (D_{ij}^z f(z)) g(z). \quad (3.4)$$

Then an expression of the form $\sum_m D_{mj}^{*,z} \Phi_m(x, z)$ with arbitrary operators $\Phi_m(x, z)$ can be added to the functional derivative without altering the equation.

Our strategy to solve this problem is the following: We introduce an algebra that is generated by symbols for the basic and derivated fields. These symbols serve as auxiliary variables. For this algebra the derivative w.r.t. a generator is defined. The polynomials in this algebra are then linearly represented as operator valued distributions acting on the Fock space. Time ordering is defined as a multi linear representation of several such polynomials as distributional Fock space operators in the next chapter. The derivative that we needed to formulate the normalization conditions occurs only in the arguments of time ordered products. With the definition of time ordered products introduced above these arguments are polynomials in the algebra. For the algebra the derivative is well defined, and therefore the normalization conditions can be formulated.

¹⁰Derivated fields means here and below fields containing a spacetime derivative.

The chapter is organized as follows: In the first section we define the algebra \mathcal{P} of auxiliary variables. In the second section the Fock space \mathcal{F} and operators therein are constructed. This construction will be completely standard and is included here to establish our notation.

In the third section the linear representation of the algebra \mathcal{P} as (distributional) Fock space operators is defined. This definition includes commutator functions for basic and derivated fields.

In the fourth section propagator functions for basic and derivated fields are constructed that have a differential operator as their inverse.

The chapter concludes with a section concerning the free model underlying Yang-Mills theory where the essential operators — ghost charge, BRS charge etc. — are defined. In particular we present Razumovs and Rybkins [RR90] proof for the positivity of that theory.

3.1. The algebra of auxiliary variables. The algebra \mathcal{P} is the graded commutative \mathbb{C} -algebra generated by auxiliary variables for the basic and derivated fields. At first we specify its generators. Therefore we determine which basic fields and which derivatives of the basic fields we wish to deal with in the model to be defined. For example, with respect to Yang-Mills theory we include Lie algebra valued vector bosons A_μ and its first derivatives $(\partial_\nu A_\mu)$, since in the interaction Yang-Mills Lagrangian both non derivated and derivated vector bosons appear. For the same reason ghosts and anti-ghosts u, \tilde{u} and their derivatives $(\partial_\nu u), (\partial_\nu \tilde{u})$ are added. If we wish to include fermionic matter, coloured spinor fields $\psi, \bar{\psi}$ must be incorporated, but no derivated spinors because these do not appear in the interaction Lagrangian. The set of fields is then completed by the double derivated vector bosons $(\partial_\nu \partial_\rho A_\mu)$ which do not appear in the Lagrangian but in the BRS current (see below). The non derivated fields are referred to as *basic fields*.

Then we define one symbol for each of these fields — with a distinct symbol for each derivative of the basic fields that is included in the list above. These symbols are the generators of \mathcal{P} . The generators corresponding to the basic fields are called the *basic generators*, those corresponding to derivated fields are called the *higher generators*. We adopt the following notation: the generators are written as φ_i where the index i numerates the basic and higher generators. Sometimes it is desirable to distinguish basic and higher generators. Then the generators are denoted as φ_i^α , where the index i numerates here the *basic* generators, and α is a multi index,

$$\alpha = (|\alpha|, \mu_1, \dots, \mu_{|\alpha|}). \quad (3.5)$$

The *degree* $|\alpha|$ of a generator φ_i^α is the number of spacetime derivatives on the corresponding field operator. Basic generators are therefore denoted as $\varphi_i^{(0)}$. The indices $\mu_1, \dots, \mu_{|\alpha|}$ are Lorentz indices corresponding to the Lorentz indices of the spacetime derivatives on the field operator. The symbols φ_i may carry additional Lorentz (e.g. if $\varphi_i = A_\mu$) or spinor (e.g. if $\varphi_i = \psi$) indices. We will define a representation of the Lorentz group on \mathcal{P} at the end of this section. To give an example for the multi indices, we relate some generators φ_i to the corresponding field operators:

$$\begin{aligned} \varphi_i^{(0)} &\longleftrightarrow \varphi_i(x), & \varphi_i^{(1,\mu)} &\longleftrightarrow \partial_x^\mu \varphi_i(x) \\ \varphi_i^{(2,\mu\nu)} &\longleftrightarrow \partial_x^\mu \partial_x^\nu \varphi_i(x) \dots \end{aligned} \quad (3.6)$$

The symbols are symmetric under permutation of the Lorentz-indices stemming from the multi-indices, e.g. $\varphi_i^{(2,\mu\nu)} = \varphi_i^{(2,\nu\mu)}$. The set \mathcal{G} of all generators of \mathcal{P} is defined as

$$\mathcal{G} \stackrel{\text{def}}{=} \{\varphi_i^\alpha : \varphi_i^\alpha \text{ has a counterpart in the desired set of fields}\}. \quad (3.7)$$

Sometimes the set of basic generators will become important:

$$\mathcal{G}_b \stackrel{\text{def}}{=} \{\varphi_i^\alpha \in \mathcal{G} : \alpha = (0)\} \subset \mathcal{G}. \quad (3.8)$$

Now \mathcal{P} is defined as the unital¹¹ algebra generated by \mathcal{G} . In addition, \mathcal{P} is graded symmetric. There are two gradings involved here: the ghost number g and the (physical) fermion number f ,

$$f, g : \{\text{monomials in } \mathcal{P}\} \rightarrow \mathbb{Z}. \quad (3.9)$$

They are additive quantum numbers,

$$g(AB) = g(A) + g(B) \quad \text{and} \quad f(AB) = f(A) + f(B) \quad \forall A, B \in \mathcal{P}, \quad (3.10)$$

and are defined as

$$\begin{aligned} g(u^\alpha) &= 1, & g(\tilde{u}^\alpha) &= -1, & g(\varphi_i^\alpha) &= 0 \quad \text{otherwise} \\ f(\psi^\alpha) &= 1, & f(\bar{\psi}^\alpha) &= -1, & f(\varphi_i^\alpha) &= 0 \quad \text{otherwise.} \end{aligned} \quad (3.11)$$

Polynomials in \mathcal{P} that have a definite ghost or fermion number are called homogeneous w.r.t. the ghost or fermion number.

Graded symmetric means that for any two elements of the algebra $A, B \in \mathcal{P}$ the following commutation relation holds:

$$AB = (-1)^{g(A)g(B)+f(A)f(B)} BA \quad \forall A, B \in \mathcal{P}. \quad (3.12)$$

This means that \mathcal{P} is the unital algebra freely generated by \mathcal{G} with the equivalence relation $AB \sim (-1)^{g(A)g(B)+f(A)f(B)} BA$ divided out. The commutation relation above implies that ghosts fulfill *commutation* relations with physical fermions.

It is important that the elements of \mathcal{P} are only symbols. In particular they are no operators in a Hilbert space and no functions on a manifold. The higher generators have no relation with the basic ones and the symbols do not satisfy field equations — e.g. $g_{\mu\nu}u^{(2,\mu\nu)} \neq 0$, where $g_{\mu\nu}$ is the metric tensor, although the ghost u is a massless Klein-Gordon field in our example. Only the representation of the symbols as operator valued distributions in Fock space will restore these relations.

On \mathcal{P} a derivative w.r.t. its generators is defined as a graded derivation according to

$$\begin{aligned} \frac{\partial}{\partial \varphi_i} (A \cdot B) &= \left(\frac{\partial A}{\partial \varphi_i} \right) \cdot B + (-1)^{f(A)f(\varphi_i)+g(A)g(\varphi_i)} A \cdot \left(\frac{\partial B}{\partial \varphi_i} \right) \\ \frac{\partial \varphi_i}{\partial \varphi_j} &= \delta_{ij} \mathbb{1} \quad \forall A, B \in \mathcal{P}, \varphi_i, \varphi_j \in \mathcal{G}. \end{aligned} \quad (3.13)$$

The representation \mathcal{R} of the Lorentz group (or its covering group $SL(2, \mathbb{C})$ for the spinors) on \mathcal{P} is defined as follows: It acts as an algebra homomorphism, i.e. a linear mapping for which

$$\mathcal{R}_\Lambda \left(\prod_i \varphi_i \right) = \prod_i \left(\mathcal{R}_\Lambda(\varphi_i) \right), \quad \Lambda \in \mathfrak{L}_+^\uparrow, \quad \varphi_i \in \mathcal{G}. \quad (3.14)$$

¹¹This means that there is an identity operator $\mathbb{1}$ included in \mathcal{P}

where \mathfrak{L}_+^\uparrow is the homogeneous proper Lorentz group. The action of \mathcal{R} on the generators is the same as for the corresponding field operators. For the basic generators this means the following: Suppose the generator $(\varphi_i)^{(0)}$ corresponds to the basic field $\varphi_i(x)$, $(\varphi_i)^{(0)} \longleftrightarrow \varphi_i(x)$, and the basic field transforms according to¹²

$$U(\Lambda)\varphi_i(x)U^{-1}(\Lambda) = \sum_j (\mathcal{R}_\Lambda)_{ij} \varphi_j(\Lambda^{-1}x), \quad \Lambda \in \mathfrak{L}_+^\uparrow \quad (3.15)$$

for some numerical matrix (\mathcal{R}_Λ) . Then the basic generator transforms according to

$$\mathcal{R}_\Lambda \left((\varphi_i)^{(0)} \right) \stackrel{\text{def}}{=} \sum_j (\mathcal{R}_\Lambda)_{ij} (\varphi_j)^{(0)} \quad (3.16)$$

with the same numerical matrix (\mathcal{R}_Λ) . In our standard example of Yang-Mills theory we have e.g.

$$\mathcal{R}_\Lambda(A^\mu) = (\Lambda)_\nu^\mu A^\nu, \quad \mathcal{R}_\Lambda(u) = u, \quad \mathcal{R}_\Lambda(\tilde{u}) = \tilde{u}, \quad \Lambda \in \mathfrak{L}_+^\uparrow \quad (3.17)$$

Here $(\Lambda)_\nu^\mu$ is the representative of Λ in the defining representation of \mathfrak{L}_+^\uparrow . The higher generators transform according to

$$\mathcal{R}_\Lambda \left((\varphi_i)^{(n, \mu_1 \dots \mu_n)} \right) \stackrel{\text{def}}{=} \sum_j \sum_{\nu_1 \dots \nu_n} (\Lambda)_{\nu_1}^{\mu_1} \dots (\Lambda)_{\nu_n}^{\mu_n} (\mathcal{R}_\Lambda)_{ij} (\varphi_j)^{(n, \nu_1 \dots \nu_n)}. \quad (3.18)$$

with (Λ) like above, and this completes the definition of \mathcal{R} .

There is also an anti-linear involution $*$ defined on \mathcal{P} . It acts on products according to

$$(aAB)^* = \bar{a}B^*A^* \quad \forall A, B \in \mathcal{P}, \quad a \in \mathbb{C}, \quad (3.19)$$

where $\bar{}$ denotes complex conjugation in \mathbb{C} .

The involution is to implement the Krein adjoint for the fields in \mathcal{P} . So take a basic generator φ_i and a basic field $\varphi_i(x)$ like above and let $(\varphi_i(x))^* = \sum_j a_{ij} \varphi_j(x)$, where the $*$ -operation on the left hand side is the Krein adjoint on the fields. Then we define for this basic generator and its corresponding higher generators

$$\left((\varphi_i)^{(n, \mu_1 \dots \mu_n)} \right)^* \stackrel{\text{def}}{=} \sum_j a_{ij} (\varphi_j)^{(n, \mu_1 \dots \mu_n)}. \quad (3.20)$$

In anticipation of the results presented in the next section we state what this means for the basic generators in the standard example:

$$(A^\mu)^* = A^\mu, \quad (u)^* = u, \quad (\tilde{u})^* = -\tilde{u} \quad (\psi)^* = \bar{\psi}\gamma^0, \quad (\bar{\psi})^* = \gamma^0\psi. \quad (3.21)$$

3.2. Fock space and Fock space operators. In this section we will construct the field operators already mentioned as operator valued distributions in the Fock space. We begin with some notations: A four-vector p on the forward light cone \overline{V}_+ will be denoted as¹³

$$\hat{p} \stackrel{\text{def}}{=} (E_p, \mathbf{p}), \quad E_p \stackrel{\text{def}}{=} \sqrt{\mathbf{p}^2}. \quad (3.22)$$

¹²The field operators and the action U of the Lorentz group on them are constructed in the next chapter

¹³The construction is outlined here for massless fields, for simplicity.

The invariant volume measure on the light cone and its Dirac distribution are defined as usual:

$$d\hat{p} \stackrel{\text{def}}{=} \frac{d^3\mathbf{p}}{2(2\pi)^3 E_p} \quad \delta(\hat{p}) \stackrel{\text{def}}{=} 2(2\pi)^3 E_p \delta(\mathbf{p}). \quad (3.23)$$

At first we must construct the Fock space \mathcal{F}_{φ_i} for each basic field that corresponds to a basic generator $\varphi_i \in \mathcal{G}_b$. That means for our standard example $\varphi_i = (A_\mu^a), (u^a, \tilde{u}_a)$ or $(\psi^r, \bar{\psi}_r)$, where a and r are possible internal indices. To this end we begin with the n -particle Hilbert space $\mathcal{H}_{\varphi_i}^n$. It is the Hilbert space of $L^2(d\hat{p}_1 \cdots d\hat{p}_n, M^n)$ functions of n momenta and n sets of indices (group-, colour- and Lorentz indices, for example) which are collectively written as a_i :

$$\varphi_{(a_1 \dots a_n)}^n(\hat{p}_1, \dots, \hat{p}_n) \in \mathcal{H}_{\varphi_i}^n \quad (3.24)$$

These functions are completely symmetric or antisymmetric under transposition of momenta and indices, $(\hat{p}_i, a_i) \leftrightarrow (\hat{p}_j, a_j)$, depending on the bosonic or fermionic character of φ_i .

The scalar product on $\mathcal{H}_{\varphi_i}^n$ is then defined as

$$(\psi^n, \phi^n) \stackrel{\text{def}}{=} \sum_{a_1 \dots a_n} \int d\hat{p}_1 \cdots d\hat{p}_n \bar{\psi}_{(a_1 \dots a_n)}^n(\hat{p}_1, \dots, \hat{p}_n) \phi_{(a_1 \dots a_n)}^n(\hat{p}_1, \dots, \hat{p}_n) \quad (3.25)$$

This scalar product is positive and allows to define a norm $\|\phi^n\| \stackrel{\text{def}}{=} (\phi^n, \phi^n)^{\frac{1}{2}}$.

With $\mathcal{H}_{\varphi_i}^0 \stackrel{\text{def}}{=} \mathbb{C}$ and $(\phi^0, \psi^0) \stackrel{\text{def}}{=} \bar{\phi}^0 \psi^0$ we can define the Fock space \mathcal{F}_{φ_i} for the field φ_i as

$$\mathcal{F}_{\varphi_i} \stackrel{\text{def}}{=} \bigoplus_{n=0}^{\infty} \mathcal{H}_{\varphi_i}^n, \quad (\phi, \psi) = \sum_{n=0}^{\infty} (\phi^n, \psi^n), \quad (3.26)$$

where \mathcal{F}_{φ_i} contains only sequences ϕ with $(\phi, \phi) < \infty$. The vector $|\omega_{\varphi_i}\rangle \stackrel{\text{def}}{=} (1, 0, 0, \dots)$ is the vacuum for this Fock space.

Next we define \mathcal{D}_{φ_i} as the dense subspace of \mathcal{F}_{φ_i} that includes only elements with a finite particle number and whose wave functions are Schwartz' test functions:

$$\phi \in \mathcal{D}_{\varphi_i} \subset \mathcal{F}_{\varphi_i} \quad \iff \quad \exists m \in \mathbb{N} : \quad \phi \in \bigoplus_{n=0}^m \mathcal{S}(M^n) \subset \mathcal{F}_{\varphi_i} \quad (3.27)$$

where $\mathcal{S}(M^n)$ is the space of Schwartz' test functions on M^n . This subspace has the advantage that Wick products are well defined operators acting on it [GW64]. It is the common domain of all operators on \mathcal{F}_{φ_i} defined below. Recently Brunetti and Fredenhagen [BF99] have found a definition of Wick products that is well posed on a bigger dense subspace than \mathcal{D}_{φ_i} , but we will stick in this thesis to the space \mathcal{D}_{φ_i} defined above.

Annihilation operators may be defined on \mathcal{D}_{φ_i} according to

$$v_a(\hat{p}) : \mathcal{D}_{\varphi_i} \rightarrow \mathcal{D}_{\varphi_i}, \quad (3.28)$$

$$[v_a(\hat{p})\phi]_{(a_1 \dots a_n)}^{(n)}(\hat{p}_1, \dots, \hat{p}_n) = \sqrt{n+1} \phi_{(a, a_1 \dots a_n)}^{(n+1)}(\hat{p}, \hat{p}_1, \dots, \hat{p}_n).$$

Their adjoint — w.r.t. the scalar product defined above — operators $v_i^+(p)$, the creation operators $v_a^+(\hat{p})$, are defined as

$$\begin{aligned} [v_a^+(\hat{p})\phi]_{(a_1\dots a_n)}^{(n)}(\hat{p}_1, \dots, \hat{p}_n) &= \\ &= \sqrt{n} \left(\delta_{a,a_1} \delta(\hat{p}_1 - \hat{p}) \phi_{(a_2\dots a_n)}^{(n-1)}(\hat{p}_2, \dots, \hat{p}_n) \right. \\ &\quad \left. \pm \sum_{k=2}^n \delta_{a,a_k} \delta(\hat{p}_k - \hat{p}) \phi_{(a_1\dots \check{a}_k\dots a_n)}^{(n-1)}(\hat{p}_2 \dots \check{p}_k \dots \hat{p}_n) \right). \end{aligned} \quad (3.29)$$

Here $\check{}$ means omission of the corresponding argument and the plus sign occurs if the field is bosonic, the minus sign if it is fermionic. The creation operators are no endomorphisms of \mathcal{D}_{φ_i} but map \mathcal{D}_{φ_i} to \mathcal{D}'_{φ_i} , the dual space of \mathcal{D}_{φ_i} . This is due to the appearance of the delta function in their definition.

Creation and annihilation operators fulfill the usual (anti-) commutation relations,

$$[v_a^+(\hat{p}), v_b(\hat{q})]_{\mp} = \delta_{ab} \delta(\hat{p} - \hat{q}), \quad [v_a^+(\hat{p}), v_b^+(\hat{q})]_{\mp} = [v_a(\hat{p}), v_b(\hat{q})]_{\mp} = 0 \quad (3.30)$$

for bosons and ghosts (where the commutator above is the graded one) and

$$\{v_a^+(\hat{p}), v_b(\hat{q})\}_{+} = \delta_{a,-b} \delta(\hat{p} - \hat{q}), \quad \{v_a^+(\hat{p}), v_b^+(\hat{q})\}_{+} = \{v_a(\hat{p}), v_b(\hat{q})\}_{+} = 0 \quad (3.31)$$

for spinors. Here $v_{-a}(\hat{p})$ is the annihilator for the field that is conjugate to the field with the annihilator $v_a(\hat{p})$.

The normal ordering — or Wick ordering — of an arbitrary product of creation and annihilation operators is defined as the same product with all the annihilation operators on the right and all the creation operators on the left. The normal product of a product W is denoted as $:W:$.

Operators on the Fock space can be defined from these distributional operators according to

$$v_a(f) = \int d\hat{p} \overline{f(\hat{p})} v_a(\hat{p}), \quad v_a^+(f) = \int d\hat{p} f(\hat{p}) v_a^+(\hat{p}). \quad (3.32)$$

With this smearing also the Wick products become operators in $\text{End}(\mathcal{D}_{\varphi_i})$.

The field operators defined below are operator valued distributions acting on the dense subspace \mathcal{D}_{φ_i} . To give a precise meaning to that expression, we define the n^{th} order operator valued distributions on an arbitrary subspace \mathcal{D} of a Fock space, abbreviated as $\text{Dist}_n(\mathcal{D})$, as \mathbb{C} -linear strongly continuous mappings

$$\text{Dist}_n(\mathcal{D}) \stackrel{\text{def}}{=} \{A : \mathcal{D}(M^n) \rightarrow \text{End}(\mathcal{D})\}. \quad (3.33)$$

where M is the Minkowski space and $\mathcal{D}M^n$ the space of test functions on M^n with compact support.

The field operators defined below are in $\text{Dist}_1(\mathcal{D}_{\varphi_i})$.

We begin the definition of the field operators that correspond to the basic generators with the vector bosons. The corresponding Fock space is denoted as \mathcal{F}_A , its dense subspace as \mathcal{D}_A . The creation and annihilation operators are denoted as $a_{\mu}^{a,+}(\hat{p})$ and $a_{\mu}^a(\hat{p})$. They fulfill the commutation relations

$$[a_{\mu}^{+,a}(\hat{p}), a_{\nu}^b(\hat{q})]_{-} = \delta^{ab} \delta_{\mu\nu} \delta(\hat{p} - \hat{q}), \quad [a_{\mu}^{+,a}(\hat{p}), a_{\nu}^{+,b}(\hat{q})]_{-} = [a_{\mu}^a(\hat{p}), a_{\nu}^b(\hat{q})]_{-} = 0. \quad (3.34)$$

The vector boson field is defined as

$$\begin{aligned} A_0^a(x) &\stackrel{\text{def}}{=} \int d\hat{p} [a_0^a(\hat{p})e^{-i\hat{p}x} - a_0^{a,+}(\hat{p})e^{i\hat{p}x}] \in \text{Dist}_1(\mathcal{D}_A), \\ A_i^a(x) &\stackrel{\text{def}}{=} \int d\hat{p} [a_i^a(\hat{p})e^{-i\hat{p}x} + a_i^{a,+}(\hat{p})e^{i\hat{p}x}] \in \text{Dist}_1(\mathcal{D}_A). \end{aligned} \quad (3.35)$$

It satisfies the commutation relation

$$[A_\mu^a(x), A_\nu^b(y)]_- = i\delta^{ab}g_{\mu\nu}D(x-y) \quad (3.36)$$

and the massless Klein-Gordon equation

$$\square^x A_\mu^a(x). \quad (3.37)$$

Here $D(x)$ is the massless Pauli-Jordan function

$$D(x) \stackrel{\text{def}}{=} 2i \int d\hat{p} \sin(\hat{p}x). \quad (3.38)$$

It has causal support. It may be split into a positive and a negative frequency part according to

$$D^+(x) \stackrel{\text{def}}{=} \int d\hat{p} e^{i\hat{p}x}, \quad D^-(x) \stackrel{\text{def}}{=} -D^+(-x). \quad (3.39)$$

Its corresponding retarded, advanced and Feynman propagators D^R, D^A and D^F are defined as

$$D^R(x) \stackrel{\text{def}}{=} \theta(x^0)D(x), \quad D^A(x) \stackrel{\text{def}}{=} -\theta(-x^0)D(x), \quad D^F(x) \stackrel{\text{def}}{=} D^R(x) - D^-(x). \quad (3.40)$$

They are the inverse of the massless Klein-Gordon operator:

$$\square^x D^{R,A,F}(x) = \delta(x). \quad (3.41)$$

Clearly D^R has retarded and D^A has advanced support.

The 0-component of the vector bosons is anti hermitian, $(A_0^a)^+ = -A_0^a$. Furthermore the scalar product is not Lorentz invariant as can be easily verified already in the one-particle space. This is a typical situation in gauge theories as described in the last chapter. To find a physical inner product on \mathcal{F}_A one must find a suitable Krein operator J_A acting on it and define

$$\langle \phi, \psi \rangle \stackrel{\text{def}}{=} (\phi, J_A \psi). \quad (3.42)$$

This suitable Krein operator is

$$J_A \stackrel{\text{def}}{=} (-1)^{N_0}, \quad N_0 = \sum_b \int d\hat{p} a_0^{b,+}(\hat{p})a_0^b(\hat{p}), \quad (3.43)$$

where N_0 is the number operator for $A^0(x)$ with eigenvalues in \mathbb{N} . It is obviously hermitian, $J = J^+$, and idempotent, $J^2 = \mathbb{1}$. With the *-involution

$$B^* \stackrel{\text{def}}{=} J_A B^+ J_A, \quad \forall B \in \text{End}(\mathcal{D}_A), \quad (3.44)$$

also called the Krein adjoint, the vector bosons become pseudo-hermitian, $(A_\mu^a)^* = A_\mu^a$. Furthermore we find the inner product $\langle \cdot, \cdot \rangle$ to define a Lorentz invariant norm, but it is indefinite.

The definition of the spinor Fock space \mathcal{F}_ψ and the field operators $\psi(x), \bar{\psi}(x)$ acting

therein proceeds in the same way and can be found in textbooks on quantum field theory. The fermions satisfy the commutation relations

$$[\psi(x), \bar{\psi}(y)]_- = -i(i\overleftarrow{\partial}_x + m)D(x-y) \quad (3.45)$$

and the field equations

$$(i\overleftarrow{\partial}_x - m)\psi = 0, \quad \bar{\psi}(-i\overleftarrow{\partial}_x - m) = 0. \quad (3.46)$$

The Krein operator J_ψ on the spinor Fock space is trivial, $J_\psi = \mathbb{1}$.

On the Fock space for the ghosts, \mathcal{F}_u with its dense subspace \mathcal{D}_u , creation and annihilation operators are denoted by $b^{a,+}(\hat{p})$, $c^{a,+}(\hat{p})$, $b^a(\hat{p})$ and $c^a(\hat{p})$, respectively. They fulfill the anti-commutation relations

$$\{b^{+,a}(\hat{p}), b^b(\hat{q})\}_+ = \delta^{ab}\delta(\hat{p}-\hat{q}), \quad \{c^{+,a}(\hat{p}), c^b(\hat{q})\}_+ = \delta^{ab}\delta(\hat{p}-\hat{q}) \quad (3.47)$$

and all other anti-commutators vanish. The ghost field $u^a(x)$ and the anti-ghost field $\tilde{u}^a(x)$ are defined as

$$\begin{aligned} u^a(x) &\stackrel{\text{def}}{=} \int d\hat{p} [b^a(\hat{p})e^{-i\hat{p}x} + c^{a,+}(\hat{p})e^{i\hat{p}x}] \in \text{Dist}_1(\mathcal{D}_u), \\ \tilde{u}^a(x) &\stackrel{\text{def}}{=} \int d\hat{p} [-c^a(\hat{p})e^{-i\hat{p}x} + b^{a,+}(\hat{p})e^{i\hat{p}x}] \in \text{Dist}_1(\mathcal{D}_u). \end{aligned} \quad (3.48)$$

Then we get for the anti-commutators of the ghosts

$$\begin{aligned} \{u^a(x), \tilde{u}^b(y)\}_+ &= -i\delta^{ab}D(x-y). \\ \{u^a(x), u^b(y)\}_+ &= \{\tilde{u}^a(x), \tilde{u}^b(y)\}_+ = 0. \end{aligned} \quad (3.49)$$

The Krein operator for the ghosts was explicitly determined by Krahe [Kra95] and reads

$$J_u = \exp\left(\frac{i\pi}{2} \int d\hat{p} [b^+(\hat{p})b(\hat{p}) - b^+(\hat{p})c(\hat{p}) + c^+(\hat{p})c(\hat{p}) - c^+(\hat{p})b(\hat{p})]\right). \quad (3.50)$$

For us it is only important that this implies for the field operators

$$(u^a(x))^* = u^a(x) \quad \text{and} \quad (\tilde{u}^a(x))^* = -\tilde{u}^a(x), \quad (3.51)$$

so the ghosts are pseudo-hermitian and the anti-ghosts are anti-pseudo-hermitian. Now we introduce the pseudo-unitary representation of the proper Poincaré group \mathfrak{P}_+^\uparrow in the individual Fock spaces. It reads for scalar fields like the ghosts

$$\begin{aligned} [U(p)\phi]^{(0)} &= \phi^{(0)}, \quad p = (a, \Lambda) \in \mathfrak{P}_+^\uparrow \\ [U(p)\phi]_{(a_1 \dots a_n)}^{(n)}(\hat{q}_1, \dots, \hat{q}_n) &= \exp(-i(\hat{q}_1 + \dots + \hat{q}_n) \cdot a) \times \\ &\quad \times \phi_{(a_1 \dots a_n)}^{(n)}(\Lambda\hat{q}_1, \dots, \Lambda\hat{q}_n). \end{aligned} \quad (3.52)$$

For vector fields like the vector bosons it reads

$$\begin{aligned} [U(p)\phi]^{(0)} &= \phi^{(0)}, \quad p = (a, \Lambda) \in \mathfrak{P}_+^\uparrow \\ ([U(p)\phi]^{(n)})_{(a_1 \dots a_n)}^{(\mu_1 \dots \mu_n)}(\hat{q}_1, \dots, \hat{q}_n) &= \exp(-i(\hat{q}_1 + \dots + \hat{q}_n) \cdot a) \times \\ &\quad \times (\Lambda)_{\nu_1}^{\mu_1} \dots (\Lambda)_{\nu_n}^{\mu_n} \left(\phi^{(n)}\right)_{(a_1 \dots a_n)}^{(\nu_1 \dots \nu_n)}(\Lambda\hat{q}_1, \dots, \Lambda\hat{q}_n) \end{aligned} \quad (3.53)$$

where the Lorentz indices μ_i have been separated from the other indices a_i and summation over repeated indices is understood. The matrices (Λ) are the representatives of Λ in the defining representation of $\mathfrak{L}_+^\uparrow \subset \mathfrak{P}_+^\uparrow$, like above. For the spinors an analogous definition holds. The vacuum vector $|\omega_{\varphi_i}\rangle$ is clearly Poincaré invariant. As was pointed out by Krahe [Kra95], it is also cyclic w.r.t. the field operators defined above.

The field operators transform according to

$$\begin{aligned} U(p)u^a(x)U^{-1}(p) &= u^a(\Lambda^{-1}x - a), & U(p)\tilde{u}^a(x)U^{-1}(p) &= \tilde{u}^a(\Lambda^{-1}x - a) \\ U(p)A_\mu^a(x)U^{-1}(p) &= (\Lambda)_\mu^\nu A_\nu^a(\Lambda^{-1}x - a). \end{aligned} \quad (3.54)$$

With the Fock spaces for the individual fields the Fock space of the entire theory \mathcal{F} , its dense subspace \mathcal{D} and the Krein operator J acting on \mathcal{F} are defined as

$$\mathcal{F} \stackrel{\text{def}}{=} \bigotimes_i \mathcal{F}_{\varphi_i} \quad \mathcal{D} \stackrel{\text{def}}{=} \bigotimes_i \mathcal{D}_{\varphi_i} \quad J \stackrel{\text{def}}{=} \bigotimes_i J_{\varphi_i}. \quad (3.55)$$

The vacuum vector of the Fock space \mathcal{F} is denoted by $|\omega\rangle$. We introduce the notation $\omega_0(A)$ for $\langle \omega | A | \omega \rangle$ for every $A \in \text{End}(\mathcal{D})$. Here $\text{End}(\mathcal{D})$ is the algebra of endomorphisms of \mathcal{D} . An important fact concerning this algebra is that it has trivial centre. Even more, for an arbitrary element $A \in \text{End}(\mathcal{D})$ the following equivalence holds:

$$\begin{aligned} [A, T(\varphi_i)(x)]_\mp &= 0 \quad \forall \varphi_i \in \mathcal{G}_b, \\ \iff A &= a \cdot \mathbb{1}, \quad a \in \mathbb{C}. \end{aligned} \quad (3.56)$$

For the proof of this assertion see Scharf [Sch95], for example.

In chapter (6) it will turn out that spacetime must be compactified in spacelike directions for the BRS charge to be a well defined operator. Therefore it is important to construct the Fock space and the operators acting on it also for a quantum field theory in the compactified spacetime. This has been done by Dütsch and Fredenhagen in [DF99, appendix A]. We refer to their results, especially concerning the choice of boundary conditions, but we do not go here into details.

3.3. The linear representation of \mathcal{P} . In this section we define the \mathbb{C} -linear representation T of the polynomials in \mathcal{P} as operator valued distributions

$$T : \mathcal{P} \rightarrow \text{Dist}_1(\mathcal{D}). \quad (3.57)$$

Linear representation means that the linear structure of \mathcal{P} is preserved, but not its structure as an algebra. This comes from the fact that a pointwise product of distributions is in general no well defined operation.

The precise definition of T will take three steps: at first it is defined for the basic generators, then for the higher generators and finally for composed elements of \mathcal{P} . The first end has already been achieved with the definition of an operator valued distribution $\varphi_i(x) \in \text{Dist}_1(\mathcal{D})$ for each basic generator $\varphi_i \in \mathcal{G}_b$. The representative of the basic generator is defined as:

$$T(\varphi_i)(x) \stackrel{\text{def}}{=} \varphi_i(x), \quad \varphi_i \in \mathcal{G}_b, \quad \varphi_i(x) \in \text{Dist}_1(\mathcal{D}). \quad (3.58)$$

This definition can work only for the basic generators since for the higher ones there are no corresponding free field operators.

For these generators we define

$$T\left((\varphi_i)^{(n, \nu_1 \dots \nu_n)}\right)(x) \stackrel{\text{def}}{=} \partial_x^{\nu_1} \dots \partial_x^{\nu_n} \varphi_i(x), \quad (\varphi_i)^{(\dots)} \in \mathcal{G}. \quad (3.59)$$

We remind the reader that there are no relations between the basic generators and the higher generators in \mathcal{P} , and that there are no field equations in \mathcal{P} . But with the definition above there is a relation established between the representatives of the basic and those of the higher generators, and the former clearly satisfy field equations. So the linear representation is not faithful.

For the representation of the composed elements in \mathcal{P} we define at first the commutator function

$$i\Delta_{ij}(x-y) = [T(\varphi_i)(x), T(\varphi_j)(y)]_{\mp}, \quad \varphi_i, \varphi_j \in \mathcal{G}. \quad (3.60)$$

Here i and j take on values also for the higher generators. With this commutator function we give an implicit definition of the representation of monomials in \mathcal{P} , namely

$$[T(W)(x), T(\varphi_i)(y)]_{\mp} = i \sum_j T \left(\frac{\partial W}{\partial \varphi_j} \right) (x) \Delta_{ij}(x-y) \quad (3.61)$$

$$\omega_0 \left(T(W)(x) \right) = 0 \quad W \in \mathcal{P}.$$

The existence of a solution is guaranteed by the observation that the normal products solve both equations. Suppose $A = \prod_i \varphi_i \in \mathcal{P}$, $\varphi_i \in \mathcal{G}$, then the normal product $:\prod_i T(\varphi_i)(x): \in \text{Dist}_1(\mathcal{D})$ is indeed a searched for solution.

The uniqueness of this solution can be seen inductively. Suppose, the representation for all monomials containing at most $k-1$ generators is defined. Then the commutator condition determines the solution for monomials of k generators up to a \mathbb{C} -number distribution — this is due to eqn. (3.56). The \mathbb{C} -number part is determined by the second condition — it is zero. So the equations above give indeed a definition for the representation of monomials in \mathcal{P} . This completes the definition of the representation.

As for the Pauli-Jordan function $D(x)$ we can find a positive and a negative frequency solution for the commutator function $\Delta_{ij}(x)$. The two point function — or positive frequency part of Δ — is denoted as Δ^+ and defined as

$$i\Delta_{ij}^+(x-y) \stackrel{\text{def}}{=} \omega_0(T(\varphi_i)(x) T(\varphi_j)(y)), \quad (3.62)$$

the negative frequency part of Δ is defined as

$$\Delta_{ij}^-(x) \stackrel{\text{def}}{=} \Delta_{ij}(x) - \Delta_{ij}^+(x). \quad (3.63)$$

3.4. The propagator functions. In this section we define propagator functions $\Delta_{ij}^R(x), \Delta_{ij}^A(x)$ analogous to $D^R(x)$ and $D^A(x)$ that are restrictions of $\Delta_{ij}(x)$ to the past and future light cone, such that $\Delta_{ij}^R(x) - \Delta_{ij}^A(x) = \Delta_{ij}(x)$. Simultaneously we search for a differential operator D_{ij}^x that takes over the part of the Klein-Gordon operator, i.e. that fulfills the equations

$$\sum_j D_{ij}^x \Delta_{jk}^{R,A}(x) = \delta_{ik} \delta(x) \implies \sum_j D_{ij}^x \Delta_{jk}(x) = 0. \quad (3.64)$$

This means in particular that the propagators must be invertible with D_{ij}^x as their inverse.

To see what form the propagators might have we take a closer look at the commutator function. If \mathcal{G} contains r generators, this is an $r \times r$ -matrix. It has the

following block diagonal structure:

$$\Delta(x) = \begin{pmatrix} \Delta^{\varphi_1}(x) & 0 & 0 & \dots \\ 0 & \Delta^{\varphi_2}(x) & 0 & \dots \\ 0 & 0 & \Delta^{\varphi_3}(x) & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \quad (3.65)$$

Here the matrices Δ^{φ_i} are of two different types. The first type corresponds to field operators that have no distinct conjugate field like the uncharged vector bosons. Then the index φ_i corresponds to the field, e.g. $\varphi_i = A_\mu$. The other type corresponds to field operators φ_i that do have such a distinct conjugate field $\tilde{\varphi}_i$ like ghosts with the anti-ghosts. In this case the field and the conjugated field form one common block in the matrix and the index φ_i corresponds to the field and its conjugated field, e.g. $\varphi_i = u, \tilde{u}$. All blocks include also the commutators of the derivatives as far as higher generators exist in \mathcal{G} that correspond to these derivatives. In our standard example it has the form

$$\Delta(x) = \begin{pmatrix} \Delta^A(x) & 0 & 0 \\ 0 & \Delta^{u, \tilde{u}}(x) & 0 \\ 0 & 0 & \Delta^{\psi, \bar{\psi}}(x) \end{pmatrix} \quad (3.66)$$

if QED is treated where no internal indices appear. In Yang-Mills theory, where internal indices do appear, there is an individual block for each index on the diagonal. From now on we will disregard internal indices for their inclusion is straightforward. The vector boson part has the form

$$\Delta^A(x) \stackrel{\text{def}}{=} g_{\mu\nu} \begin{pmatrix} D(x) & -\partial_x^{\nu_1} D(x) & \partial_x^{\nu_1} \partial_x^{\nu_2} D(x) \\ \partial_x^{\rho_1} D(x) & -\partial_x^{\nu_1} \partial_x^{\rho_1} D(x) & \partial_x^{\nu_1} \partial_x^{\rho_1} \partial_x^{\nu_2} D(x) \\ \partial_x^{\rho_2} \partial_x^{\rho_1} D(x) & -\partial_x^{\nu_1} \partial_x^{\rho_2} \partial_x^{\rho_1} D(x) & \partial_x^{\nu_1} \partial_x^{\rho_2} \partial_x^{\rho_1} \partial_x^{\nu_2} D(x) \end{pmatrix}, \quad (3.67)$$

the ghost part

$$\Delta^{u, \tilde{u}}(x) \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 0 & -D(x) & \partial_x^{\nu_1} D(x) \\ 0 & 0 & -\partial_x^{\rho_1} D(x) & \partial_x^{\rho_1} \partial_x^{\nu_1} D(x) \\ D(x) & -\partial_x^{\nu_1} D(x) & 0 & 0 \\ \partial_x^{\rho_1} D(x) & -\partial_x^{\rho_1} \partial_x^{\nu_1} D(x) & 0 & 0 \end{pmatrix} \quad (3.68)$$

and the spinor part

$$\Delta^{\psi, \bar{\psi}}(x) \stackrel{\text{def}}{=} \begin{pmatrix} 0 & -i(i\partial + m)D_m(x) \\ i(i\partial - m)D_m(x) & 0 \end{pmatrix}. \quad (3.69)$$

The distribution D_m is the Pauli-Jordan function for mass m . The matrices are given here in the basis

$$\left((A_\mu)^{(0)}, (A_\mu)^{(1,\nu_1)}, (A_\mu)^{(2,\nu_1\nu_2)} \right)^t \quad (3.70)$$

for the vector bosons,

$$\left((u)^{(0)}, (u)^{(1,\nu_1)}, (\tilde{u})^{(0)}, (\tilde{u})^{(1,\nu_1)} \right)^t \quad (3.71)$$

for the ghosts and anti-ghosts and

$$\left((\psi)^{(0)}, (\bar{\psi})^{(0)} \right)^t, \quad (3.72)$$

for the spinors, where t denotes transposition.

The natural attempt would be to replace the Pauli-Jordan function $D(x)$ by its retarded, $D^R(x)$, or advanced, $D^A(x)$, propagator in each entry to define the matrices $\Delta_{ij}^R(x)$ and $\Delta_{ij}^A(x)$. These would clearly be well defined distributions with the desired support properties, but they would not be invertible. This comes from the fact that with this definition each row would be the derivative of the row above, and therefore the determinant — w.r.t. convolution — of these matrices would vanish. To improve the definition above we observe that the matrices $\Delta_{ij}^{R,A}(x)$ are defined by their desired support properties — $\text{supp } \Delta_{ij}^R(x) \subset \overline{V}_+$ and $\text{supp } \Delta_{ij}^A(x) \subset \overline{V}_-$ — and their relation to the commutator function everywhere but in the origin. That means that we may alter the propagator functions only at the origin, i.e. by delta distributions or its derivatives at the individual entries. As a further restriction of possible propagators we demand that this modification does not increase the scaling degree (see below) of the individual entries and that it does not change the Lorentz transformation property of that entry.

Scaling degree means the following: For every numerical distribution d one can define a dilated distribution

$$d_\lambda(x) = d(\lambda x) \quad \lambda \in \mathbb{R}^+ \setminus \{0\}. \quad (3.73)$$

Clearly d_λ is a numerical distribution, too. Then the *scaling degree* $\text{sd}(d)$ of d w.r.t. the origin is defined, according to Steinmann [Ste71], as

$$\text{sd}(d) \stackrel{\text{def}}{=} \inf \left\{ \beta \in \mathbb{R} : \lim_{\lambda \searrow 0} \lambda^\beta d_\lambda = 0 \right\}, \quad (3.74)$$

where the equation in the bracket holds in the sense of distributions.

The restriction on the scaling degree fixes some entries uniquely, e.g. $\Delta_{ij}^R(x) = D^R(x)$ if $\varphi_i = u$ and $\varphi_j = \tilde{u}$. For others there remains a certain ambiguity, e.g.

$$\Delta_{ij}^R(x) = -g_{\mu\nu} \partial^\rho \partial^\sigma D^R(x) - C g_{\mu\nu} g^{\rho\sigma} \delta(x) \quad (3.75)$$

if $\varphi_i = (A_\mu)^{(1,\rho)}$ and $\varphi_j = (A_\nu)^{(1,\sigma)}$. The numerical constant C is then arbitrary. In the following we define propagator functions with an inverse that is a differential operator and we will give later the explicit form of these differential operators.

The propagators have the same block diagonal structure as the commutator function:

$$\Delta^{R,A}(x) = \begin{pmatrix} \Delta_{R,A}^{\varphi_1}(x) & 0 & 0 & \dots \\ 0 & \Delta_{R,A}^{\varphi_2}(x) & 0 & \dots \\ 0 & 0 & \Delta_{R,A}^{\varphi_3}(x) & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \quad (3.76)$$

In the following we will consider only the construction of the retarded propagator. The advanced propagator is defined as $\Delta^A = \Delta^R - \Delta$. For the determination of the individual blocks we notice that usually the $(0,0)$ -component¹⁴ of the commutator function has a scaling degree smaller than the spacetime dimension, so that its retarded solution is uniquely determined by the following condition:

$$\Delta_{00}^{R,\varphi_i}(x) = \Delta_{00}^{\varphi_i}(x) \quad x \notin \overline{V}_+, \quad \text{supp } \Delta_{00}^R \in \overline{V}_+. \quad (3.77)$$

With this the general matrix element of a block $\Delta_R^{\varphi_i}$ of the retarded propagator can be written as

$$\Delta_{jk}^{R,\varphi_i}(x) = (-1)^k \partial^{\mu_1} \dots \partial^{\mu_k} \partial^{\nu_1} \dots \partial^{\nu_j} \Delta_{00}^{R,\varphi_i}(x) + (-1)^k \delta_{jk} C_{\varphi_i,k} \delta(x). \quad (3.78)$$

(no summation over k in the last term). The constants $C_{\varphi_i,k}$ are non zero real numbers, $C_{\varphi_i,k} \in \mathbb{R} \setminus \{0\}$. As these constants will determine the normalization of higher order time ordered products (c.f. next chapter), they will be called normalization constants.

In our standard example the propagator has the form

$$\Delta^R(x) = \begin{pmatrix} \Delta_R^A(x) & 0 & 0 \\ 0 & \Delta_R^{u,\tilde{u}}(x) & 0 \\ 0 & 0 & \Delta_R^{\psi,\bar{\psi}}(x) \end{pmatrix}. \quad (3.79)$$

For the vector boson block of the retarded propagator, Δ_R^A , we omit all spacetime arguments because it otherwise would not fit into the line. Then it reads

$$\Delta_R^A = g_{\mu\nu} \begin{pmatrix} D^R & -\partial^\nu D^R & \partial^\nu \partial^\rho D^R \\ \partial^\sigma D^R & -\partial^\nu \partial^\sigma D^R - C_{A,1} g^{\nu\sigma} \delta & \partial^\nu \partial^\rho \partial^\sigma D^R \\ \partial^\sigma \partial^\tau D^R & -\partial^\nu \partial^\sigma \partial^\tau D^R & \partial^\nu \partial^\rho \partial^\sigma \partial^\tau D^R - C_{A,2} g^{\nu\sigma} g^{\rho\tau} \delta \end{pmatrix}, \quad (3.80)$$

For the ghosts we get the contribution,

$$\Delta_R^{u,\tilde{u}}(x) = \begin{pmatrix} 0 & -d_R^u(x) \\ d_R^u(x) & 0 \end{pmatrix} \quad (3.81)$$

¹⁴We start the numbering of columns and rows with zero, such that the index of a column or row agrees with the degree of the corresponding generator

with the 2×2 -matrices

$$d_R^u(x) = \begin{pmatrix} D^R(x) & -\partial_x^{\nu_1} D^R(x) \\ \partial_x^{\rho_1} D^R(x) & -\partial_x^{\nu_1} \partial_x^{\rho_1} D^R(x) - C_{u,1} g^{\nu_1 \rho_1} \delta(x) \end{pmatrix}. \quad (3.82)$$

The spinors finally give

$$\Delta_R^{\psi, \bar{\psi}}(x) = \begin{pmatrix} 0 & -(i\partial\!\!\!/ + m)D_m^R(x) \\ (i\partial\!\!\!/ - m)D_m^R(x) & 0 \end{pmatrix} \quad (3.83)$$

D_m^R is the retarded part of D_m . All the matrices are given in the same basis as for the commutator function. The retarded propagator function has obviously retarded support and agrees with the commutator function outside the forward light cone. The advanced propagator $\Delta^A = \Delta^R - \Delta$ has advanced support and agrees with the commutator function outside the backward light cone. In the example above the respective advanced propagators can be derived from the retarded propagators by a substitution of D^R with D^A .

We define also a Feynman propagator

$$\Delta^F : \quad \Delta_{ij}^F(x) \stackrel{\text{def}}{=} \Delta_{ij}^R(x) - \Delta_{ij}^-(x). \quad (3.84)$$

Now we come to the differential operator valued matrix D^x that inverts the propagators defined above, i.e. for which the equation

$$\sum_j D_{ij}^x \Delta_{jk}^{R,A,F}(x) = \delta_{ik} \delta(x) \quad (3.85)$$

holds. It is an $r \times r$ matrix, where r was the number of generators in \mathcal{G} . It has the usual block diagonal form:

$$D^x = \begin{pmatrix} D^{\varphi_1, x} & 0 & 0 & \dots \\ 0 & D^{\varphi_2, x} & 0 & \dots \\ 0 & 0 & D^{\varphi_3, x} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad (3.86)$$

or, in our standard example,

$$D^x = \begin{pmatrix} D^{A, x} & 0 & 0 \\ 0 & D^{u, \bar{u}, x} & 0 \\ 0 & 0 & D^{\psi, \bar{\psi}, x} \end{pmatrix}. \quad (3.87)$$

Like for the propagators the individual blocks correspond to field operators or pairs of conjugated fields. We define the blocks for single fields as $(s+1) \times (s+1)$ -matrices if higher generators up to degree s are included in \mathcal{G} for that field. Let $K^{\varphi_i, x}$ be the differential operator that defines the field equation for $\varphi_i(x)$, i.e. which fulfills the equation

$$K^{\varphi_i, x} \Delta_R^{\varphi_i}(x) = \delta(x), \quad (3.88)$$

e.g. $K^{A,x} = \square^x$. Then the corresponding block is written in the basis

$$\left((\varphi_i)^{(0)}, (\varphi_i)^{(1,\nu_1)}, \dots, (\varphi_i)^{(s,\nu_1 \dots \nu_s)} \right)^t \quad (3.89)$$

as the matrix with the components

$$\begin{aligned} D_{00}^{\varphi_i,x} &= \left(K^{\varphi_i,x} - \sum_{k=1}^n (-1)^k C_{\varphi_i,k}^{-1} \square^k \right) \\ D_{0k}^{\varphi_i,x} &= (-1)^k C_{\varphi_i,k}^{-1} (\partial^{\nu_1} \dots \partial^{\nu_k}) \\ D_{j0}^{\varphi_i,x} &= C_{\varphi_i,j}^{-1} (\partial^{\sigma_1} \dots \partial^{\sigma_j}) \\ D_{kk}^{\varphi_i,x} &= -C_{\varphi_i,k}^{-1} (g^{\nu_1 \sigma_1} \dots g^{\nu_k \sigma_k}) \\ D_{jk}^{\varphi_i,x} &= 0 \quad \text{otherwise.} \end{aligned} \quad (3.90)$$

where the constants $C_{\varphi_i,k}$ are those determined in the propagator functions. Again we exemplify the definition above for our standard example. The only uncharged fields there are the vector bosons. In the same basis as for the commutator function, the block $D^{A,x}$ has the form

$$D^A \stackrel{\text{def}}{=} \begin{pmatrix} (1 + C_{A,1}^{-1})\square - C_{A,2}^{-1}\square^2 & -C_{A,1}^{-1}\partial^{\nu_1} & C_{A,2}^{-1}\partial^{\nu_1}\partial^{\nu_2} \\ C_{A,1}^{-1}\partial^{\rho_1} & -C_{A,1}^{-1}g^{\rho_1\nu_1} & 0 \\ C_{A,2}^{-1}\partial^{\rho_1}\partial^{\rho_2} & 0 & -C_{A,2}^{-1}g^{\rho_1\nu_1}g^{\rho_2\nu_2} \end{pmatrix}. \quad (3.91)$$

For the charged fields we construct according to the rules above one block $D^{\varphi_i,x}$ for the fields φ_i and one block $D^{\tilde{\varphi}_i,x}$ for the conjugated fields $\tilde{\varphi}_i$. Like for the propagators, the combined block for the fields and conjugated fields reads then

$$D^{\varphi_i,\tilde{\varphi}_i,x} \stackrel{\text{def}}{=} \begin{pmatrix} 0 & D^{\tilde{\varphi}_i,x} \\ -D^{\varphi_i,x} & 0 \end{pmatrix} \quad (3.92)$$

in the basis

$$\left((\varphi_i)^{(0)}, \dots, (\varphi_i)^{(s,\nu_1 \dots \nu_s)}, (\tilde{\varphi}_i)^{(0)}, \dots, (\tilde{\varphi}_i)^{(s,\nu_1 \dots \nu_s)} \right)^t, \quad (3.93)$$

if higher generators up to degree s are included. The expressions for our standard example, i.e. for the ghosts and the spinors, read then

$$D^{u,\tilde{u}} \stackrel{\text{def}}{=} \frac{1}{C_{u,1}} \begin{pmatrix} 0 & 0 & (1 + C_{u,1})\square & -\partial^{\nu_1} \\ 0 & 0 & \partial^{\rho_1} & -g^{\nu_1\rho_1} \\ -(1 + C_{u,1})\square & \partial^{\nu_1} & 0 & 0 \\ -\partial^{\rho_1} & g^{\nu_1\rho_1} & 0 & 0 \end{pmatrix}. \quad (3.94)$$

For the spinors no higher generators are included in our example, so they contribute the expression

$$D^\psi \stackrel{\text{def}}{=} \begin{pmatrix} 0 & -(i\partial + m) \\ (i\partial - m) & 0 \end{pmatrix}. \quad (3.95)$$

where the operator in the second line acts from the right.

It is easily verified by direct calculation that this differential operator really inverts the propagators. Furthermore, the representatives of the generators satisfy the following *free field equations*:

$$\sum_j D_{ij}^x T(\varphi_j)(x) = 0. \quad (3.96)$$

Here the sum runs over *all* generators. This equation holds independently of the choice of the normalization constants $C_{\varphi_i, k}$. From its definition it is already clear that the commutator function is annihilated by D^x :

$$\sum_j D_{ij}^x \Delta_{jk}(x) = 0. \quad (3.97)$$

If D^x is determined, the propagator functions Δ^R, Δ^A and Δ^F are *uniquely* determined by the following conditions:

- $\Delta^{R,A,F}(x)$ must fulfil eqn. (3.85)
- $\Delta^R(x) = \Delta(x) \quad \forall x \notin \bar{V}_-$ and $\Delta^R(x) = 0 \quad \forall x \in \bar{V}_- \setminus \{0\}$
- $\Delta^A(x) = \Delta^R(x) - \Delta(x)$
- $\Delta^F(x) = \Delta^R(x) - \Delta^+(x)$.

So D^x is a relativistically covariant, hyperbolic differential operator with a unique solution for the Cauchy problem. In particular the normalization constants $C_{\varphi_i, k}$ that appear in the propagators are uniquely determined by their choice in the differential operator D^x .

We do not claim that our choice for the operator D^x or the propagators is the most general one. But we point out that there are serious restrictions to the choice of the propagators. As we already saw, the apparently easiest choice is not invertible, and all other choices we tried proved to be invertible, but with pseudodifferential operators as their inverse instead of differential operators. We do not examine the question whether field equations with pseudodifferential operators are suitable choices within the general framework of quantum field theory. Instead we stick to differential operators as one is used to, the more so as the propagators we have defined above are completely sufficient for our purposes.

3.5. The free BRS theory. We examine in this section the free theory that includes vector bosons, spinors and ghosts. This is the theory that served as an example throughout the considerations above. The generators for the algebra \mathcal{P} are in this model $A_\mu^a, (A_\mu^a)^{(1, \nu)}$ and $(A_\mu^a)^{(2, \nu \rho)}$ for the Lie algebra valued vector bosons, $u^a, \tilde{u}^a, (u^a)^{(1, \mu)}$ and $(\tilde{u}^a)^{(1, \mu)}$ for the respective ghosts and anti-ghosts and ψ^r and $\bar{\psi}^r$ for the coloured spinors. The field operators that correspond to the generators $A_\mu^a, u^a, \tilde{u}^a, \psi^r$ and $\bar{\psi}^r$ are already constructed as operators in the Fock space \mathcal{F} with a common dense domain \mathcal{D} . As we already mentioned when we constructed the Fock space, the inner product $\langle \cdot, \cdot \rangle$ is indefinite. To perform the BRS construction, we must define a BRS charge and a ghost charge and prove that the state space is

positive.

At first we define the ghost current

$$k^\mu \stackrel{\text{def}}{=} i \sum_a \left[(u^a)^{(0)} (\tilde{u}^a)^{(1,\mu)} - (u^a)^{(1,\mu)} (\tilde{u}^a)^{(0)} \right] \in \mathcal{P}. \quad (3.98)$$

and the BRS current

$$j_B^\mu \stackrel{\text{def}}{=} \sum_a \left[(u^a)^{(1,\mu)} (A_\nu^a)^{(1,\nu)} - (u^a)^{(0)} (A_\nu^a)^{(2,\nu\mu)} \right] \in \mathcal{P}. \quad (3.99)$$

as elements of \mathcal{P} . Then their definitions as operators in the Fock space follow immediately as

$$k^\mu(x) = T(k^\mu)(x) \quad \text{and} \quad j_B^\mu(x) = T(j_B^\mu)(x). \quad (3.100)$$

Taking into account the field equations, we note that both operators are conserved,

$$\partial_\mu^x k^\mu(x) = \partial_\mu^x j_B^\mu(x) = 0. \quad (3.101)$$

Now it is possible to define the corresponding charges, the ghost charge Q_c and the BRS charge Q_B , as

$$Q_c \stackrel{\text{def}}{=} \lim_{\lambda \searrow 0} \int d^4x h_\lambda(x) k^0(x) \quad \text{and} \quad Q_B \stackrel{\text{def}}{=} \lim_{\lambda \searrow 0} \int d^4x h_\lambda(x) j_B^0(x). \quad (3.102)$$

Here $h_\lambda \in \mathcal{D}(M)$, $\lambda \in \mathbb{R}^+ \setminus \{0\}$ is a test function that has the following structure:

$$h_\lambda(x) = \lambda h^t(\lambda \cdot x_0) b(\lambda \mathbf{x}), \quad h^t \in \mathcal{D}(\mathbb{R}) \quad b \in \mathcal{D}(\mathbb{R}^3), \quad (3.103)$$

$$\int dx_0 h^t(x_0) = 1,$$

with $b = 1$ on an open domain including the origin of \mathbb{R}^3 . Due to a general argument of Requardt [Req76] the limit $\lambda \searrow 0$ exists and it is independent of the choice of h_λ . So the charges define well posed operators in the Fock space. The charges have no counterpart in the symbolic algebra, because the integrals would make no sense there.

The ghost transformation and the BRS transformation are (anti-) derivations on the algebra $\text{End}(\mathcal{D})$:

$$s_c(A) \stackrel{\text{def}}{=} [Q_c, A]_-, \quad \text{and} \quad s_0(A) \stackrel{\text{def}}{=} [Q_B, A]_{\mp} \quad \forall A \in \text{End}(\mathcal{D}). \quad (3.104)$$

The derivations give for the basic fields the following results:

$$\begin{aligned} s_c(u^a(x)) &= u^a(x), & s_c(\tilde{u}^a(x)) &= -\tilde{u}^a(x), \\ s_c(\varphi_i(x)) &= 0 \quad \text{otherwise}, \\ s_0(A_\mu^a(x)) &= i\partial_\mu^x u^a(x), & s_0(\tilde{u}^a(x)) &= -i\partial_x^\mu A_\mu^a(x), \\ s_0(\varphi_i(x)) &= 0 \quad \text{otherwise}. \end{aligned} \quad (3.105)$$

Finally we must prove that for the physical state space, defined as the state cohomology of \mathcal{F} w.r.t. the BRS charge Q_B above, the positivity assumption holds. This has already been done by Kugo and Ojima [KO79], but we present here a modern version that is due to Razumov and Rybkin [RR90]. We collect here only the essential points of their proof.

At first they note that the entire space \mathcal{D} can be decomposed as

$$\mathcal{F} = \text{im } Q_B \oplus (\text{im } Q_B \cap \text{im } Q_B^+) \oplus \text{im } Q_B^+ \quad (3.106)$$

with

$$\begin{aligned} \text{im } Q_B \oplus (\text{im } Q_B \cap \text{im } Q_B^+) &= \ker Q_B \\ \text{and } (\text{im } Q_B \cap \text{im } Q_B^+) \oplus \text{im } Q_B^+ &= \ker Q_B^+. \end{aligned} \quad (3.107)$$

Then they propose an alternative definition of the physical (pre-) Hilbert space according to

$$\mathcal{H}_{\text{phys}} = (\text{im } Q_B \cap \text{im } Q_B^+) \quad (3.108)$$

or, which is the same,

$$\mathcal{H}_{\text{phys}} = \ker \left(\{Q_B, Q_B^+\}_+ \right). \quad (3.109)$$

This definition of the physical (pre-) Hilbert space deviates from the original one in the way that it selects from each equivalence class there exactly one representative. Now a direct calculation of the operator $\{Q_B, Q_B^+\}_+$ reveals

$$\{Q_B, Q_B^+\}_+ = N_0 + N_L + N_g \quad (3.110)$$

where N_0 is the number operator of scalar vector bosons introduced above, N_L is the corresponding operator for the longitudinal vector bosons and N_g the operator that counts the total number ghosts and anti-ghosts. Comparison with the definition of the Krein operator

$$J = (-1)^{N_0} \otimes \mathbb{1} \otimes J_g \quad (3.111)$$

reveals that $J = \mathbb{1}$ on $\mathcal{H}_{\text{phys}} = \ker \left(\{Q_B, Q_B^+\}_+ \right)$. Therefore the inner product must be positive on $\mathcal{H}_{\text{phys}}$ since the original scalar product was. It is not necessary to restrict the physical Hilbert space to the kernel of Q_c since this Hilbert space is already contained in $\ker N_g \subset \ker Q_c$.

The result ensuring positivity holds also for our definition of $\mathcal{H}_{\text{phys}}$ as

$$\mathcal{H}_{\text{phys}} = \overline{(\ker Q_B, \mathcal{D}) / (\text{im } Q_B, \mathcal{D})}^{\|\cdot\|}, \quad (3.112)$$

since in this definition each equivalence class modulo $(\text{im } Q_B)$ corresponds to exactly one element of $\ker \left(\{Q_B, Q_B^+\}_+ \right)$, and the inner product does not depend on the choice of the representative within the equivalence class.

Then the algebra of observables is defined as usual,

$$\mathcal{A}_{\text{ph}} \stackrel{\text{def}}{=} \left((\ker s, \text{End}(\mathcal{D})) \cap (\ker s_c, \text{End}(\mathcal{D})) \right) / (\text{im } s, \text{End}(\mathcal{D})). \quad (3.113)$$

As was pointed out by Dütsch and Fredenhagen, the algebra is faithfully represented in the physical Hilbert space.

4. TIME ORDERED PRODUCTS AND THEIR NORMALIZATION

In this chapter the construction of time ordered products, antichronological products and their respective properties are presented. Since the construction is in general not unique, normalization conditions are postulated that restrict the ambiguity. The construction of time ordered products is the central point for causal perturbation theory, which is presented in the next chapter. In particular this is the point where renormalization takes place in this framework.

The time ordering of n arbitrary Wick polynomials $W_1(x_1), \dots, W_n(x_n)$, $W_i \in \text{Dist}_1(\mathcal{D})$, can be done by the following prescription

$$T(W_1(x_1) \cdots W_n(x_n)) \stackrel{\text{def}}{=} (-1)^{f(\pi)+g(\pi)} \sum_{\pi \in \mathcal{P}_{\underline{n}}} \theta(x_{\pi(1)}^0 - x_{\pi(2)}^0) \cdots \cdots \theta(x_{\pi(n-1)}^0 - x_{\pi(n)}^0) W_{\pi(1)}(x_{\pi(1)}) \cdots W_{\pi(n)}(x_{\pi(n)}) \quad (4.1)$$

if all the points x_i are different. Here $\mathcal{P}_{\underline{n}}$ is the set of permutations of $\underline{n} \stackrel{\text{def}}{=} \{1, \dots, n\}$, $f(\pi)$ is the number of transpositions in $\pi \in \mathcal{P}_{\underline{n}}$ that involve arguments with an odd fermion number and $g(\pi)$ is the number of those that involve arguments with odd ghost number. θ is the Heaviside step function,

$$\theta(x) = \begin{cases} 1 & \text{if } x^0 > 0 \\ 0 & \text{otherwise.} \end{cases} \quad (4.2)$$

The crucial point is that this prescription is not defined for coinciding points, because the Wick polynomials W_i are distributions that “do not like to be multiplied by discontinuous functions” [Sto93]. This is the origin of the ultraviolet divergences of quantum field theories. The prescription above gives, as it stands, well defined distributions only on a smaller space of test functions than $\mathcal{D}(M^n)$. This is the space of test functions in $\mathcal{D}(M^n)$ that vanish with all their derivatives if two or more of their spacetime arguments coincide. To form time ordered products these distributions on the smaller space of test functions must be extended to elements of $\text{Dist}_n(\mathcal{D})$.

The time ordering of n arguments is usually regarded as a mapping of n operator valued distributions in $\text{Dist}_1(\mathcal{D})$ to an operator valued distribution in $\text{Dist}_n(\mathcal{D})$. We however define the time ordering of n arguments as a mapping of n polynomials in \mathcal{P} to an operator valued distribution in $\text{Dist}_n(\mathcal{D})$. As already mentioned this has the advantage that the normalization conditions can be formulated also for derivated fields. Beside that technical point the extension of the distributions follows the method of Epstein and Glaser [EG73]. The extension exists always but is in general not unique. Therefore for each combination of arguments one element in $\text{Dist}_n(\mathcal{D})$ must be chosen as the time ordered product of these arguments. This choice is called the *normalization* of that time ordered product according to Scharf [Sch95].

The normalization conditions implement various properties of the time ordered products that are desired from the physical point of view. The postulation of the normalization conditions restricts the number of possible normalizations, but the extension is in general still not unique.

This chapter is organized as follows: The first section presents the properties of time ordered products that are required for their construction. In the next section this construction is performed. Antichronological products are defined in the third

section. The chapter concludes with a section in which the normalization conditions are formulated.

4.1. Properties of time ordered products. The construction of time ordered products proceeds by induction. The time ordered products of a number of arguments are built out of the time ordered products with fewer arguments. This construction works only if the time ordered products with fewer arguments have certain properties. These properties are presented here. They are

P1 (Well posedness): The time ordering operator for n arguments, T_n , is a multi linear mapping of n polynomials in \mathcal{P} to the operator valued distributions of order n on the dense subspace $\mathcal{D} \subset \mathcal{F}$:

$$T_n : \underbrace{\mathcal{P} \times \cdots \times \mathcal{P}}_{n \text{ times}} \rightarrow \text{Dist}_n(\mathcal{D}). \quad (4.3)$$

If the arguments are explicitly given, the index n indicating the number of arguments will be omitted.

From the physical interpretation of time ordering we would expect that the time ordering operator must have at least two arguments, for otherwise there is nothing to be put in order. But it turns out to be useful to extend the mapping defined above formally also to the cases $n = 0$ and $n = 1$. This is achieved by the following definitions:

$$T_0 \stackrel{\text{def}}{=} \mathbb{1}, \quad \mathbb{1} \in \text{End}(\mathcal{D}) \quad (4.4)$$

and

$$T_1(W)(x) \stackrel{\text{def}}{=} T(W)(x), \quad \forall W \in \mathcal{P}. \quad (4.5)$$

Here T on the right hand side is the linear representation defined in the last chapter. The operator valued distributions obtained by the time ordering are called *time ordered products* or *T-products*. The time ordered product of the polynomials W_1, \dots, W_n is written as

$$T(W_1, \dots, W_n)(x_1, \dots, x_n). \quad (4.6)$$

The definition above implies that the tensor product of two time ordered products with m and n arguments is a well defined operator valued distribution in $\text{Dist}_{m+n}(\mathcal{D})$. Arguments that are multiples of the identity can be removed according to

$$T(W_1, \dots, W_n, a \cdot \mathbb{1})(x_1, \dots, x_n, y) = a \cdot T(W_1, \dots, W_n)(x_1, \dots, x_n) \quad \forall a \in \mathbb{C}. \quad (4.7)$$

P2 (Graded symmetry): Time ordered products are *totally graded symmetric* under permutations of their indices. That means

$$\begin{aligned} & T(W_{\pi(1)}, \dots, W_{\pi(n)})(x_{\pi(1)}, \dots, x_{\pi(n)}) \\ &= (-1)^{f(\pi)+g(\pi)} T(W_1, \dots, W_n)(x_1, \dots, x_n) \quad \forall \pi \in \mathcal{P}_n, \end{aligned} \quad (4.8)$$

where the integers $f(\pi)$, $g(\pi)$ were defined in eqn. (4.1).

P3 (Causality): Time ordered products are *causal*, that means they fulfill eqn. (4.1) for non coinciding points. Even more, outside the total diagonal Diag_n (see below) the time ordered product of n arguments is completely determined by those

that have fewer arguments. The total diagonal $\text{Diag}_n \subset M^n$ is the set where *all* points coincide:

$$\text{Diag}_n = \{(x_1, \dots, x_n) \in M^n : x_1 = \dots = x_n\}. \quad (4.9)$$

If not all points x_i coincide there exists a spacelike surface $\Sigma \subset M$ that separates the points $X = \{x_1, \dots, x_n\}$ into a future subset Z and a past subset $Z^c = X \setminus Z$ such that

$$\Sigma \cap X = \emptyset, \quad Z \subset (\Sigma + \bar{V}_+), \quad Y \subset (\Sigma + \bar{V}_-). \quad (4.10)$$

This situation will be denoted as $Z \gtrsim Z^c$. Furthermore we introduce the abbreviation

$$T(W_Z)(x_Z) \stackrel{\text{def}}{=} T(W_1, \dots, W_k)(x_1, \dots, x_k) \quad \text{if } Z = \{x_1, \dots, x_k\}. \quad (4.11)$$

Causality means that the time ordered product $T(W_X)(x_X)$ is required to satisfy *causal factorization*:

$$T(W_X)(x_X) = T(W_Z)(x_Z)T(W_{Z^c})(x_{Z^c}) \quad \text{if } Z \gtrsim Z^c. \quad (4.12)$$

It provides a recursive definition of the time ordered products up to the diagonal Diag_n . There the separation into future and past subsets is impossible and therefore no causal factorization exists. Validity of causal factorization for every number of arguments implies that spacelike separated time ordered products (anti-)commute¹⁵:

$$[T(W_Z)(x_Z), T(W_{Z^c})(x_{Z^c})]_{\mp} = 0 \quad \text{if } Z \not\gtrsim Z^c. \quad (4.13)$$

P4 (Translational invariance): Time ordered products are *translationally invariant*, that means that for every $a \in M$ the following equation holds:

$$\begin{aligned} (\text{Ad } U(p)) T(W_1, \dots, W_n)(x_1, \dots, x_n) &= \\ &= T(W_1, \dots, W_n)(x_1 - a, \dots, x_n - a) \quad \forall p = (a, \mathbb{1}) \in \mathcal{P}_+^{\uparrow}. \end{aligned} \quad (4.14)$$

Here U is the representation of the Poincaré group in the Fock space introduced in the last chapter.

4.2. Inductive construction of time ordered products. In this section the inductive construction of the time ordered products is outlined. It goes back to Epstein and Glaser [EG73]. We use a formulation of their procedure proposed by Stora [Sto93] and recently elaborated by Brunetti and Fredenhagen in [BF99]. This section will not contain the proofs of the theorems. For them we refer to the latter article.

Formally the time ordering is also defined for a single argument by the linear representation T . The latter is uniquely defined for all $W \in \mathcal{P}$. This will serve as a starting point for the induction. Obviously the representation satisfies properties **P1** - **P4**.

We suppose that all T -products for up to $n - 1$ arguments are already constructed and satisfy properties **P1** - **P4**. Due to property **P3** the time ordered products for n arguments are therefore completely determined on $M^n \setminus \text{Diag}_n$, i.e. for all test

¹⁵The notation $Z \not\gtrsim Z^c$ means that Z and Z^c are spacelike separated, i.e. $Z \gtrsim Z^c$ and $Z^c \gtrsim Z$.

functions in $\mathcal{D}(M^n \setminus \text{Diag}_n)$. To construct the distributions off the diagonal we introduce at first a partition of $M^n \setminus \text{Diag}_n$ into the spaces

$$\mathfrak{C}_Z \stackrel{\text{def}}{=} \{(x_1, \dots, x_n) \in M^n : x_i \notin (x_j + \overline{V}_-), \quad \forall i \in Z, j \in Z^c\} \quad (4.15)$$

for every $Z \neq \emptyset, Z \neq X$.

It is easy to see (and has been proven in [BF99, Lemma 4.1]) that

$$\bigcup_{\substack{Z \neq \emptyset, \\ Z \neq X}} \mathfrak{C}_Z = M^n \setminus \text{Diag}_n. \quad (4.16)$$

Furthermore we define

$$T_Z(W_X)(x_X) \stackrel{\text{def}}{=} \begin{cases} T(W_Z)(x_Z)T(W_{Z^c})(x_{Z^c}) & \text{if } (x_1, \dots, x_n) \in \mathfrak{C}_Z, \\ 0 & \text{otherwise.} \end{cases} \quad (4.17)$$

$T_Z(W_X)(x_X)$ is a well defined operator valued distribution in $\text{Dist}_n(\mathcal{D})$. Finally we choose an arbitrary locally finite C^∞ -partition of unity for $M^n \setminus \text{Diag}_n$,

$$\{f_Z\} : \quad \sum_Z f_Z = 1 \text{ on } M^n \setminus \text{Diag}_n, \quad (4.18)$$

$\text{supp } f_Z \in \mathfrak{C}_Z, \quad f_Z \in C^\infty(M^n \setminus \text{Diag}_n).$

The restriction of $T(W_X)(x_X)$ to $M^n \setminus \text{Diag}_n$ ¹⁶ can now be defined as

$$T^0(W_X)(x_X) \stackrel{\text{def}}{=} \sum_Z f_Z(x_X) \cdot T_Z(W_X)(x_X). \quad (4.19)$$

This definition does not depend on the choice of $\{f_Z\}$ because we assumed that eqns. (4.12) and (4.13) hold for the T -products with fewer arguments. This makes the T^0 -products well defined operator valued distributions on test functions in $\mathcal{D}(M^n \setminus \text{Diag}_n)$ that satisfy the properties **P1** - **P4**. For the proofs see [BF99].

For the construction of the time ordered products with n arguments the T^0 -products must be extended to the diagonal. They are linear combinations of products of numerical distributions t^0 with Wick products : $W_1(x_1) \cdots W_n(x_n)$;, where the $W_i(x_i)$ are Wick monomials in $\text{Dist}_1(\mathcal{D})$. It is not trivial that these products exist, because distributions are multiplied at the same spacetime point, but it was shown by Epstein and Glaser that translational invariance implies that this product is indeed well defined — this result is referred to as “Theorem 0” in [EG73, p. 229]. From the modern point of view the product exists because the wave front sets of the distributions do not linearly combine to zero in the cotangent spaces, see [BF99]. For the extension of the operator valued distributions to the diagonal it suffices to extend each numerical distribution t^0 and to prove that the resulting product is well defined. The latter is no problem here because the “Theorem 0” applies also to the extended distributions.

Translational invariance (**P4**) implies that the numerical distributions t^0 depend only on the relative coordinates $(y_1, \dots, y_{n-1}) \stackrel{\text{def}}{=} (x_1 - x_n, \dots, x_{n-1} - x_n)$ such that Diag_n is the origin in the space of the y 's. This allows us to give a further restriction to the extension of the numerical distributions to the diagonal: Each distribution t^0 is regarded as a distribution in the space of relative coordinates. Then the scaling degree of the extended distribution t must not exceed that of the original distribution t^0 in relative coordinates. Brunetti and Fredenhagen [BF99]

¹⁶That means $T^0(W_X)(x_X) : \mathcal{D}(M^n \setminus \text{Diag}_n) \rightarrow \text{End}(\mathcal{D})$

prove that such an extension always exists as a well defined distribution for test functions in $\mathcal{D}(M^{n-1})$ — or in $\mathcal{D}(M^n)$ if one returns to the original coordinates. It is unique only if the original distribution has a scaling degree $\text{sd}(t^0)$ that satisfies the following inequality:

$$\text{sd}(t^0) < (n - 1) \times d \quad (4.20)$$

where $n - 1$ is the number of relative coordinates and d the spacetime dimension. This can be seen as follows: The distribution t is already determined up to the diagonal Diag_n . In other words, two extensions may differ only by a delta distribution with support at the origin of the relative coordinates or by a derivative of it. If the scaling degree of t^0 satisfies the inequality above, it is not possible to add a delta distribution or a derivative of it without violating the restriction on the scaling degree. Therefore the solution is unique then. In general the inequality does not hold and the extension is therefore ambiguous, corresponding to the freedom of finite renormalization in other renormalization procedures.

4.3. Antichronological products. In this section we define antichronological products. This definition can be given recursively as $\overline{T}_0 = \mathbb{1}$ and¹⁷

$$\begin{aligned} \overline{T}(W_X)(x_X) &\stackrel{\text{def}}{=} \\ &= - \sum_{Y \subset X, Y \neq \emptyset} (-1)^{|Y|} T(W_Y)(x_Y) \overline{T}(W_{Y^c})(x_{Y^c}) \\ &= - \sum_{Y \subset X, Y \neq X} (-1)^{|Y^c|} \overline{T}(W_Y)(x_Y) T(W_{Y^c})(x_{Y^c}) \end{aligned} \quad (4.21)$$

for $n \geq 1$. Here possible signs that come from changes in the order of the arguments are neglected for simplicity. They can be easily recovered using **P2**, which holds for the \overline{T} -products, too (see below).

Iterating the recursive definition above one finds the following explicit expression for the \overline{T} -products:

$$\overline{T}(W_X)(x_X) = \sum_P (-1)^{|P|+|X|} \prod_{p \in P} T(W_p)(x_p). \quad (4.22)$$

Here the sum runs over all partitions P of X into $|P|$ nonempty subsets. With this definition the antichronological products become for non coinciding points, i.e. $x_i \neq x_j \ \forall i \neq j$,

$$\begin{aligned} \overline{T}(W_X)(x_X) &= \sum_{\pi \in \mathcal{P}_X} \theta(x_{\pi(1)}^0 - x_{\pi(2)}^0) \cdots \\ &\quad \cdots \theta(x_{\pi(n-1)}^0 - x_{\pi(n)}^0) T(W_{\pi(n)})(x_{\pi(n)}) \cdots T(W_{\pi(1)})(x_{\pi(1)}). \end{aligned} \quad (4.23)$$

The antichronological products satisfy properties **P1**, **P2** and **P4**. Property **P3** holds for them in the reverse order. That means that under the same conditions and with the same notation as in **P3** antichronological products satisfy

$$\overline{T}(W_X)(x_X) = T(W_{Z^c})(x_{Z^c}) T(W_Z)(x_Z), \quad Z \succsim Z^c \quad (\mathbf{P3}')$$

justifying their name since they are defined like the time ordered products but with the opposite order.

¹⁷The notation is the same as in eqn. (4.11)

4.4. Normalization conditions. In this section we formulate the normalization conditions that restrict the ambiguity in the extension of the \overline{T} -products to the diagonal. They implement Poincaré covariance **(N1)** and unitarity **(N2)**. They define the time ordered products up to a \mathbb{C} -number distribution **(N3)** and determine them uniquely if at least one argument is a generator from \mathcal{G} **(N4)**. Finally they determine Ward identities for the ghost current **(N5)** and the BRS current **(N6)**. It is proven that the conditions **(N1)** - **(N5)** have common solutions. For condition **(N6)** this must be done for the individual models.

The first normalization condition establishes Poincaré covariance w.r.t. the representation U of the Poincaré group \mathcal{P}_+^\uparrow introduced in chapter (3). It reads

$$\begin{aligned} (\text{Ad } U(p)) T(W_1, \dots, W_n)(x_1, \dots, x_n) &= \\ &= T(\mathcal{R}_\Lambda(W_1), \dots, \mathcal{R}_\Lambda(W_n))(\Lambda^{-1}x_1 - a, \dots, \Lambda^{-1}x_n - a) \end{aligned} \quad (\text{N1})$$

for every $p = (a, \Lambda) \in \mathcal{P}_+^\uparrow$ and all monomials $W_i \in \mathcal{P}$. Here \mathcal{R}_Λ is the representation of the Lorentz group on \mathcal{P} introduced in section (3.1). Property **P4** is in view of **(N1)** only the special case with $p = (a, \mathbb{1})$.

Popineau and Stora [PS82] have proven that this condition has always a solution, but their article is unfortunately not published. So we refer the reader to Scharf [Sch95, p. 282] for the proof. Recently Prange, Bresser and Pinter, [BPP99] and [Pra99], have found even a general construction prescription for covariant normalizations.

The second normalization condition establishes pseudo-unitarity by means of

$$T(W_1, \dots, W_n)^*(x_1, \dots, x_n) = \overline{T}(W_n^*, \dots, W_1^*)(x_n, \dots, x_1) \quad \forall W_i \in \mathcal{P}, \quad (\text{N2})$$

where the *-involution on the left hand side is the Krein adjoint on $\text{End}(\mathcal{D})$, while the *-involution on the right hand side is the adjoint operation in \mathcal{P} defined in section (3.1). Note that the order of the arguments is reversed. It can of be put into the original order by means of **P2**.

It was already shown by Epstein and Glaser [EG73] that eqn. **(N2)** can always be accomplished. Their argument and the compatibility of **(N2)** with **(N1)** can be easily understood: Suppose, **(N2)** holds for all integers $m < n$ simultaneously with eqn. **(N1)**. Then for every normalization $T' = T(W_1, \dots, W_n)$ that is compatible with eqn. **(N1)** the distribution $T = \frac{1}{2}(T' + T'^*)$ satisfies eqn. **(N2)** and will also be an extension of T^0 because **(N2)** holds for the T^0 -products by induction. It will automatically be a solution of eqn. **(N1)** since the representation U was chosen to be pseudo-unitary, i.e. $U(p)^* = U(p)^{-1}$.

To formulate the third normalization condition we remind the reader of the commutator function $\Delta_{jk}(x)$, eqn. (3.60) in section [3.3]. The normalization condition reads:

$$\begin{aligned} \left[T(W_1, \dots, W_n)(x_1, \dots, x_n), \varphi_i(y) \right]_{\mp} &= \\ &= i \sum_{k=1}^n \sum_j \Delta_{ij}(x_k - y) \cdot T\left(W_1, \dots, \frac{\partial W_k}{\partial \varphi_j}, \dots, W_n\right)(x_1, \dots, x_n), \end{aligned} \quad (\text{N3})$$

for every $W_i \in \mathcal{P}$, $\varphi_i(y) = T(\varphi_i)(y)$, $\varphi_i \in \mathcal{G}$. The second sum runs over all generators in \mathcal{G} , not only the basic generators.

Since an element of $\text{End}(\mathcal{D})$ is a multiple of the identity if it (anti-) commutes with all the $\varphi_i(y)$ — see eqn. (3.56) in section (3.2) —, this condition determines

the time ordered products uniquely up to a \mathbb{C} -number, provided the time ordered products that involve the sub monomials are known. This can be explicitly seen in an equivalent equation, the *causal Wick expansion*

$$T(W_1, \dots, W_n)(x_1, \dots, x_n) = \sum_{\gamma_1, \dots, \gamma_n} \omega_0 \left(T(W_1^{(\gamma_1)}, \dots, W_n^{(\gamma_n)})(x_1, \dots, x_n) \right) \times \frac{:\varphi^{\gamma_1}(x_1) \cdots \varphi^{\gamma_n}(x_n):}{\gamma_1! \cdots \gamma_n!}. \quad (4.24)$$

Here the $\gamma_i \in \mathbb{N}^r$ are multi indices, vectors with one entry for each of the r generators in \mathcal{G} , i.e.

$$\gamma_i = ((\gamma_i)_1, \dots, (\gamma_i)_r) \in \mathbb{N}^r \quad (4.25)$$

The $W^{(\gamma_i)}$ are derivatives,

$$W^{(\gamma_i)} \stackrel{\text{def}}{=} \frac{\partial^{|\gamma_i|} W}{\partial^{(\gamma_i)_1} \varphi_1 \cdots \partial^{(\gamma_i)_r} \varphi_r}, \quad (4.26)$$

where $|\gamma_i| = \sum_{k=1}^r (\gamma_i)_k$. The φ^{γ_i} are defined as

$$\varphi^{\gamma_i}(x) \stackrel{\text{def}}{=} T \left(\prod_{k=1}^r \varphi_k^{(\gamma_i)_k} \right) (x). \quad (4.27)$$

Finally

$$(\gamma_i)! \stackrel{\text{def}}{=} \prod_{k=1}^r (\gamma_i)_k!. \quad (4.28)$$

It is shown in the appendix, section A.1, that the causal Wick expansion is indeed equivalent with **(N3)**. Compatibility with eqn. **(N1)** is easily verified since **(N3)** respects the Poincaré transformation properties. With the same construction as after eqn. **(N2)** one can show that for every common solution of **(N1)** and **(N3)** a normalization can be constructed that is also a solution of **(N2)**.

In particular in the formulation (4.24) of **(N3)** it is immediately clear that only $\omega_0(T(W_1, \dots, W_n)(x_1, \dots, x_n))$ — the term with $\gamma_1 = \dots = \gamma_n = 0$ in (4.24) — is left open to be normalized, since all other terms are determined by the time ordered products for the sub monomials. These distributions correspond to the vacuum diagrams of the respective time ordered product in the Feynman graph picture. So condition **(N3)** has the consequence that only vacuum diagrams need to be (re-)normalized, a fact that is well known from other renormalization procedures.

The fourth normalization condition is a differential equation that uniquely determines time ordered products with at least one generator $\varphi_i \in \mathcal{G}$ among its arguments. This assertion holds under the assumption that the time ordered products for fewer arguments are already known. The condition reads:

$$\begin{aligned} \sum_j D_{ij}^y T(W_1, \dots, W_n, \varphi_j)(x_1, \dots, x_n, y) = \\ = i \sum_{k=1}^n T \left(W_1, \dots, \frac{\partial W_k}{\partial \varphi_i}, \dots, W_n \right) (x_1, \dots, x_n) \delta(x_k - y), \end{aligned} \quad (\text{N4})$$

where $W_i \in \mathcal{P}$, $\varphi_j \in \mathcal{G}$. It is proven in the appendix, section A.2, that condition **(N4)** has common solutions with condition **(N3)**. Compatibility with condition

(N1) is again immediate since (N4) is Poincaré covariant. A solution of (N1), (N3) and (N4) that satisfies also (N2) can be found by the same procedure as above.

In the chapter concerning the interacting theory we will see that eqn. (N4) already implies the interacting field equations.

Condition (N4) possesses an alternative formulation, like (N3). Its integrated version reads

$$\begin{aligned}
& T(W_1, \dots, W_n, \varphi_i)(x_1, \dots, x_n, y) = \\
& = i \sum_{k=1}^n \sum_j \Delta_{ij}^F(y - x_k) T\left(W_1, \dots, \frac{\partial W_k}{\partial \varphi_j}, \dots, W_n\right)(x_1, \dots, x_n) \\
& + \sum_{\gamma_1 \dots \gamma_n} \omega_0 \left(T(W_1^{(\gamma_1)}, \dots, W_n^{(\gamma_n)})(x_1, \dots, x_n) \right) \frac{\varphi^{\gamma_1}(x_1) \dots \varphi^{\gamma_n}(x_n) \varphi_i(y)}{\gamma_1! \dots \gamma_n!}.
\end{aligned} \tag{4.29}$$

The sum over j runs again over all generators including the higher ones.

This formulation shows explicitly that with eqn. (N4) the time ordered products with at least one generator among its arguments are already determined. In appendix (A.2) the equivalence of the two formulations is proven.

Eqn. (N4) uniquely fixes the Feynman propagators for derivated fields. These in turn determine all tree level diagrams. Comparing (N4) with results from other renormalization procedures shows an important difference between the causal approach and other approaches: The definition of the propagators for the derivated fields differ between the causal approach and other approaches. Therefore also the Green's functions at tree level are different. The difference between the conventional propagators and our prescription is labelled by the normalization constants $C_{\varphi_i, k}$. Only if all these constants are set to zero the difference disappears. But we saw already that the propagators are then no longer invertible. For example, in the conventional renormalization procedures we have

$$\omega_0 \left(T(\partial_x^\mu A_\nu(x), \partial_y^\nu A_\rho(y)) \right) = -i \partial_\rho^x \partial_x^\mu D^F(x - y), \tag{4.30}$$

while the corresponding propagator in our causal theory reads

$$\omega_0 \left(T((A_\nu)^{1, \mu}, (A_\rho)^{1, \nu})(x, y) \right) = -i \partial_\rho^x \partial_x^\mu D^F(x - y) - i C_{A, 1} \delta_\rho^\mu \delta(x - y). \tag{4.31}$$

Now we come to the Ward identities for the ghost current. This is a normalization condition for time ordered products that contain a ghost current k^μ — see section (3.5) — as an argument. It reads

$$\begin{aligned}
& \partial_\mu^y T(W_1, \dots, W_n, k^\mu)(x_1, \dots, x_n, y) = \\
& = \sum_{k=1}^n g(W_k) \delta(y - x_k) T(W_1, \dots, W_n)(x_1, \dots, x_n)
\end{aligned} \tag{N5}$$

for all monomials $W_i \in \mathcal{P}$. It holds if none of the arguments contains a generator $(u^a)^{(\alpha)}$ or $(\tilde{u}^a)^{(\alpha)}$ with $|\alpha| \geq 2$.

The proof that this normalization condition has common solutions with the conditions (N1) - (N4) is given in appendix (B.1). For a technical reason that will be explained there this normalization condition can only be proven for arguments W_i that do not contain k^μ as a sub monomial — in particular k^μ itself is excluded. In the examples where (N5) is applied in the following chapters this limitation will

not be relevant.

We state here one particular fact that will come out in the proof: There exists exactly one choice for the normalization constant $C_{u,1}$ such that condition **(N5)** has common solutions with **(N1)** - **(N4)**. This choice is $C_{u,1} = -1$.

Following Dütsch and Fredenhagen [DF99] who made the calculation for the Ward identities for the electric current (see below) we prove in appendix (B.1) that there exists an integrated version of **(N5)**, namely

$$\begin{aligned} s_c T(W_1, \dots, W_n)(x_1, \dots, x_n) &= \\ &= \left(\sum_{k=1}^n g(W_k) \right) T(W_1, \dots, W_n)(x_1, \dots, x_n). \end{aligned} \quad (4.32)$$

So as a consequence of **(N5)** the ghost number of a time ordered product is simply the sum of the ghost numbers of its arguments.

Eqn. **(N5)** and (4.32) are equivalent in the following sense: If **(N5)** holds then (4.32) is automatically valid, too. If (4.32) holds, then a normalization can be found that is compatible with **(N5)**. For details see appendix (B.1).

Dütsch and Fredenhagen [DF99] proved an analogous Ward identity for the electric current $j_{\text{el}}^\mu = \bar{\psi} \gamma^\mu \psi$. Here ψ and $\bar{\psi}$ are the electron and the positron field, respectively. Their Ward identity reads in our language

$$\begin{aligned} \partial_\mu^y T(W_1, \dots, W_n, j_{\text{el}}^\mu)(x_1, \dots, x_n, y) &= \\ &= i \left(\sum_{k=1}^n f(W_k) \delta(y - x_k) \right) T(W_1, \dots, W_n)(x_1, \dots, x_n). \end{aligned} \quad (\mathbf{N5}')$$

It holds if the monomials W_i do not contain generators $\psi^{(\alpha)}$ or $\bar{\psi}^{(\alpha)}$ with $|\alpha| \geq 1$. The existence of common solutions of **(N5')** with the other normalization conditions can be proven along the same lines as for the ghost current Ward identities, provided that either none of the arguments W_i contains j_{el}^μ as a sub monomial or that all the W_i are the QED Lagrangian $\mathcal{L}_{QED} = A_\mu j_{\text{el}}^\mu$ or sub monomials of it.

To formulate the Ward identity for the BRS current we anticipate here a condition for the Lagrangian that will be illuminated more closely in section (5.4). In QED and Yang-Mills theory there exist so called $Q(n)$ -vertices for the Lagrangians. These are polynomials $\mathcal{L}_1^\mu, \mathcal{L}_2^{\mu\rho}, \dots \in \mathcal{P}$ totally antisymmetric in their Lorentz-indices for which the following identities hold:

$$s_c T(\mathcal{L}_i^{\mu_1, \dots, \mu_i})(x) = i \partial_\rho^x T(\mathcal{L}_{i+1}^{\mu_1, \dots, \mu_i, \rho})(x). \quad (4.33)$$

We admit only polynomials \mathcal{L} as Lagrangians if there exist such $Q(n)$ -vertices and in addition so called $R(n)$ -vertices $\mathcal{M}_1, \mathcal{M}_2, \dots$ that are polynomials in \mathcal{P} which satisfy the following condition: There exists a normalization of $T(\mathcal{L}_i^{\mu_1, \dots, \mu_i}, j^\mu)$ that is compatible with the normalization conditions **(N1)** - **(N4)** and for which the equation

$$\begin{aligned} \partial_\mu^y T(\mathcal{L}_i^{\mu_1, \dots, \mu_i}, j^\mu)(x, y) &= i \partial_\nu^x \left(\delta(x - y) \mathcal{L}_{i+1}^{\mu_1, \dots, \mu_i, \nu}(x) \right) \\ &\quad + i \left(\partial_\nu^x \delta(x - y) \right) \mathcal{M}_{i+1}^{\mu_1, \dots, \mu_i, \nu}(x) \end{aligned} \quad (4.34)$$

holds. The series of equations terminates at a certain point i.e. there exists an $m \in \mathbb{N}$ with $\mathcal{L}_m = 0, \mathcal{M}_m = 0$. This is the condition **(C4)** in section (5.4). The $R(n)$ -vertices are totally antisymmetric in their Lorentz indices, too.

With the notion of $Q(n)$ -vertices and the $R(n)$ -vertices we can give the next normalization condition, the Ward identities for the BRS current:

$$\begin{aligned} \partial_\mu^y T(\mathcal{L}_{i_1}, \dots, \mathcal{L}_{i_n}, j^\mu)(x_1, \dots, x_n, y) &= \\ &= i \sum_{k=1}^n \partial_\nu^k \left(\delta(y - x_k) T(\mathcal{L}_{i_1}, \dots, \mathcal{L}_{i_{k+1}}^\nu, \dots, \mathcal{L}_{i_n})(x_1, \dots, x_n) \right) \\ &\quad + i \sum_{k=1}^n \left(\partial_\nu^k \delta(y - x_k) \right) T(\mathcal{L}_{i_1}, \dots, \mathcal{M}_{i_{k+1}}^\nu, \dots, \mathcal{L}_{i_n})(x_1, \dots, x_n) \end{aligned} \quad (\mathbf{N6})$$

where $i \in \mathbb{N}$ and we define $\mathcal{L}_0 = \mathcal{L}$.

The same calculation leading to eqn. (4.32) can also be applied to condition (N6) and gives the *generalized operator gauge invariance*

$$\begin{aligned} s_0 T(\mathcal{L}_{i_1}, \dots, \mathcal{L}_{i_n})(x_1, \dots, x_n) &= \\ &= i \sum_{k=1}^n \partial_\nu^k T(\mathcal{L}_{i_1}, \dots, \mathcal{L}_{i_{k+1}}^\nu, \dots, \mathcal{L}_{i_n})(x_1, \dots, x_n). \end{aligned} \quad (4.35)$$

Dütsch, Hurth, Krahe and Scharf, [DHKS94a] - [DHS95b], found that for eqn. (4.35) to hold in Yang-Mills theory for two arguments the normalization constant $C_{A,1}$ in eqn. (3.91) must be $C_{A,1} = -\frac{1}{2}$.

Unfortunately there exists no general proof that condition (N6) can always be accomplished or that it is compatible with (N3) and (N4)¹⁸. But we can show that the generalized operator gauge invariance together with (N5) is already sufficient for (N6). For the construction of solutions of (N6) under the assumption that generalized operator gauge invariance holds see appendix (B.2).

The proof that either the eqn. (N6) or eqn. (4.35) have common solutions with the other normalization conditions must be done in individual models.

As far as we know QED is the only example where this is done — for the proof see section (7.1). The existence of solutions for eqn. (4.35) is in QED a direct consequence of the existence of solutions for the electric current Ward identities (N5').

In Yang-Mills theories the solutions for eqn. (4.35) can be explicitly given in first order, see section (7.2). A detailed study of (4.35) with $i_1 = \dots = i_n = 0$ for Yang-Mills theory without matter fields can be found in [DHKS94a] - [DHS95b] — this equation is called *operator gauge invariance*. This result has been generalized to Yang-Mills theory with matter fields by Dütsch [Düt96]. They come to the result that operator gauge invariance holds in that theory provided a weak assumption concerning the infrared behaviour is satisfied. Loosely speaking the infrared behaviour must not be too bad. It is usually assumed that this assumption is satisfied, otherwise not even off shell Green's functions would exist.

It should be possible to prove the generalized version of operator gauge invariance under the same assumption and along the same lines as in their calculation, but this has not been done up to now — and it is probably a long winded work, the

¹⁸At a first sight it may seem that (N4) has nothing to do with (N6) since there is no generator in the time ordered products whose normalization (N6) determines. The point is that compatibility with (N3) requires a set of relations among which are also some that involve time ordered products that contain a generator. Then (N4) could fix their normalization in a way that compatibility between (N3) and (N6) is inhibited. In this sense we think that (N4) and (N6) shall be compatible.

original calculation took a series of four articles.

Another promising strategy to prove generalized operator gauge invariance is to translate the results of algebraic renormalization [PS95] to causal perturbation theory. The *descent equations* can be viewed as the generalized operator gauge invariance version of that framework. It has been proven in [PS95] that they can be accomplished for Yang-Mills theories. Unlike our causal approach algebraic renormalization is a loop expansion, i.e. an expansion in the parameter \hbar and not in the coupling constant. Furthermore it is a functional approach, in contrast to causal perturbation theory which is an operator approach. So in order to make the results cited above available to the causal theory some translational work has to be performed. This has not been done up to now.

For the equations **(N6)** with $\sum i_n \geq 5$ the compatibility of normalization conditions is easy to prove: These T -products comply automatically with **(N3)** and **(N4)** since their ghost number, which is the minimal number of field operators in the Wick products in the causal Wick expansion, exceeds the spacetime dimension, so we are in the situation of the inequality (4.20) and therefore the extension is unique and complies with **(N3)**, **(N4)** and **(N5)**.

We have stated altogether six normalization conditions for the time ordered products (where for the last it remains open whether it can always be accomplished). One could ask whether these conditions suffice to make the extension of the T^0 -products to the diagonal unique. Unfortunately this is not true. There remains a certain ambiguity, even though calculations in first order show that the normalization conditions restrict the freedom of the extensions severely — in fact there are many examples where the above conditions suffice to make the extension unique.

The decisive point is that the normalization conditions suffice to prove a lot of relations in the interacting theory like field equations, nilpotency of the interacting BRS charge and others, notwithstanding the remaining ambiguity.

Another interesting feature of these normalization conditions is that a subsequent enlargement of the algebra \mathcal{P} — by the introduction of new basic fields or by inclusion of generators for higher derivatives of the basic fields than before — does not change the normalization of the time ordered products with arguments in the original, smaller algebra. Moreover these normalizations do not depend on the model with regard to which they are considered. For example there are certain time ordered products that occur both in QED and in Yang-Mills theory, but due to our construction their normalization is the same in both cases, provided the normalization constants $C_{\varphi_i, k}$ are chosen equal. This is of course a consequence of the fact that the normalizations are completely independent of the Lagrangian. The latter is in this context a polynomial in \mathcal{P} not outstanding from the others. So the idea behind the whole construction is to determine all time ordered products a priori, store them in a big library and fetch them if they are needed for a certain calculation. The remaining ambiguity of the time ordered products is certainly a handicap. Ambiguous time ordered products should be laid down in this library with an endorsement that they are ambiguous and what the allowed normalizations are.

5. LOCAL CAUSAL PERTURBATION THEORY

This chapter is devoted to the formulation of local causal perturbation theory. It will establish the connection of time ordered products with interacting quantum field theory. In the framework of causal perturbation theory the S-matrix and the interacting field operators are defined in terms of time ordered products, see below. As usual the interaction will be defined by the S-matrix. But as we investigate local theories, there will be no interpretation of the S-matrix available as an operator mapping in-states onto out-states. These asymptotic states are a global concept that loses its meaning in a local framework. Nevertheless the S-matrix is the central object of the interacting theory. It determines the theory since the local interacting field operators are defined in terms of it.

In the causal approach infrared divergences are completely independent of the ultraviolet divergences — in particular there cannot be a cancellation of infrared with ultraviolet divergences. We circumvent the problem of infrared divergences by considering only local theories. By a local theory we mean the following situation: We choose an open, bounded domain $\mathcal{O} \subset M$ in Minkowski space — usually such that it is causally complete — in which the interacting fields are localized and consider the field algebra generated by these fields.

The crucial observation that makes it possible to abandon the adiabatic limit and therefore to avoid infrared divergences is due to Brunetti and Fredenhagen [BF97]. They found that a modification of the interaction outside the domain \mathcal{O} induces only a unitary transformation of the field algebra. Since this does not touch the physical content of the theory, it is in particular possible to switch off the interaction outside \mathcal{O} . With the coupling being a test function, infrared divergences cannot occur.

The chapter gives a short presentation of causal perturbation theory in the formulation of Epstein and Glaser [EG73]. For the reader interested in details of the causal approach we refer to the textbook of Scharf [Sch95]. We use here the notation of Epstein and Glaser which is different from that in the book of Scharf.

At first we construct the S-matrix by means of time ordered products. In the second section we define interacting field operators in terms of retarded products. Advanced and causal products are defined in the third section. The model we consider is determined by an interaction Lagrangian. It is a polynomial in \mathcal{P} , but not every polynomial in \mathcal{P} can serve as a Lagrangian. We postulate in the fourth section five conditions such a polynomial must satisfy in order to define a possible Lagrangian.

5.1. The S-matrix. The S-matrix is defined as a formal power series in terms of time ordered products of the Lagrangian as

$$S(g\mathcal{L}) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^4x_1 \cdots d^4x_n g(x_1) \cdots g(x_n) T(\mathcal{L}, \dots, \mathcal{L})(x_1, \dots, x_n) \quad (5.1)$$

Here g is the coupling “constant”, i.e. in our approach a real test function in $\mathcal{D}(M)$. The notation $(g\mathcal{L})$ in the argument of S is of course only symbolic, since the product of a test function in $\mathcal{D}(M)$ with a symbol in \mathcal{P} is not defined. It means that the polynomials are the arguments of the time ordering which are smeared out with

the test functions. For sums the symbolic notation means e.g.

$$S(g_1\mathcal{W}_1 + g_2\mathcal{W}_2) \stackrel{\text{def}}{=} \mathbb{1} + i \int d^4x_1 [g_1(x_1)T(\mathcal{W}_1)(x_1) + g_2(x_1)T(\mathcal{W}_2)(x_1)] + \dots \quad (5.2)$$

The S-matrix is an element in $\tilde{\mathbf{C}} \cdot \text{End } \mathcal{D}$, i.e. the set of formal power series whose elements are endomorphisms on \mathcal{D} . This is true because $T(\mathcal{L}, \dots, \mathcal{L})(x_1, \dots, x_n) \in \text{Dist}_n(\mathcal{D})$ and $g(x_1) \cdots g(x_n) \in \mathcal{D}(M^n)$.

The S-matrix is also the generating functional of the time ordered products, i.e. the time ordered products can be recovered from the S-matrix by means of

$$T(W_1, \dots, W_n)(x_1, \dots, x_n) = \frac{\delta^n}{i^n \delta g_1(x_1) \cdots \delta g_n(x_n)} S\left(\sum_{k=1}^n g_k W_k\right) \Big|_{g_1 = \dots = g_n = 0} \quad (5.3)$$

The inverse S-matrix $S^{-1}(g\mathcal{L})$ is also a formal power series. From eqn. (4.21) we conclude

$$S^{-1}(g\mathcal{L}) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int d^4x_1 \cdots d^4x_n g(x_1) \cdots g(x_n) \bar{T}(\mathcal{L}, \dots, \mathcal{L})(x_1, \dots, x_n). \quad (5.4)$$

The S-matrix is also pseudo unitary, $S(g\mathcal{L})^* = S^{-1}(g\mathcal{L})$, by means of normalization condition (N2).

5.2. Interacting fields and retarded products. The interacting fields are constructed according to Bogoliubov as operator valued distributions by

$$(W_i)_{\text{int}}^{g\mathcal{L}}(y) \stackrel{\text{def}}{=} S(g\mathcal{L})^{-1} \frac{\delta}{i\delta h(y)} S(g\mathcal{L} + hW_i) \Big|_{h=0}. \quad (5.5)$$

Here h is a test function in $\mathcal{D}(M)$. The corresponding localized field operators are

$$(W_i)_{\text{int}}^{g\mathcal{L}}(f) \stackrel{\text{def}}{=} \int d^4y f(y) (W_i)_{\text{int}}^{g\mathcal{L}}(y) \quad (5.6)$$

where f is a test function with support in the domain \mathcal{O} as. The algebra of field operators that are localized in \mathcal{O} is denoted as $\tilde{\mathcal{F}}(\mathcal{O})$.

Inserting the definition of the S-matrix the distributional field operators can be written as

$$(W_i)_{\text{int}}^{g\mathcal{L}}(y) = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^4x_1 \cdots d^4x_n g(x_1) \cdots g(x_n) \times R(\mathcal{L}, \dots, \mathcal{L}; W_i)(x_1, \dots, x_n; y). \quad (5.7)$$

This expression contains the so called retarded or R -products whose definition in terms of T - and \bar{T} -products reads

$$R(W_1, \dots, W_n; W_i)(x_1, \dots, x_n; y) \stackrel{\text{def}}{=} \sum_{Y \subset X} (-1)^{|Y|} \bar{T}(W_Y)(x_Y) T(W_{Y^c}, W_i)(x_{Y^c}, y). \quad (5.8)$$

Here $X = \{x_1, \dots, x_n\}$. For the notation we refer to eqn. (4.11).

According to eqn. (4.21) the retarded products can be alternatively expressed as

$$\begin{aligned} R(W_1, \dots, W_n; W_i)(x_1, \dots, x_n; y) \\ = \sum_{Y \in X} (-1)^{|Y|} \bar{T}(W_Y, W_i)(x_Y, y) T(W_{Y^c})(x_{Y^c}). \end{aligned} \quad (5.9)$$

Causality (4.13) implies that the retarded products have retarded support (justifying their name), i.e.

$$\begin{aligned} \text{supp } R(W_X; W_i)(x_X, y) \\ \subset \{(x_1, \dots, x_n, y) \in M^{n+1} : x_i \in (y + \bar{V}_-) \quad \forall x_i \in X\}. \end{aligned} \quad (5.10)$$

The interacting fields in $\tilde{\mathcal{F}}(\mathcal{O})$ therefore depend only on the interaction in the past of \mathcal{O} . From the definition of the interacting field distributions Dütsch and Fredenhagen derive in [DF99] the commutator relation:

$$\begin{aligned} \left[(W^1)_{\text{int}}^{g\mathcal{L}}(x), (W^2)_{\text{int}}^{g\mathcal{L}}(y) \right]_{\mp} &= - \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^4 x_1 \cdots d^4 x_n g(x_1) \cdots g(x_n) \times \\ &\quad \left\{ R(\mathcal{L}, \dots, \mathcal{L}, W^1; W^2)(x_1, \dots, x_n, x; y) \right. \\ &\quad \left. \mp R(\mathcal{L}, \dots, \mathcal{L}, W^2; W^1)(x_1, \dots, x_n, y; x) \right\}. \end{aligned} \quad (5.11)$$

5.3. The advanced and the causal product. The advanced product is defined as

$$\begin{aligned} A(W_1, \dots, W_n; W_i)(x_1, \dots, x_n; y) \\ \stackrel{\text{def}}{=} \sum_{Y \subset X} (-1)^{|Y|} T(W_{Y^c})(x_{Y^c}) \bar{T}(W_Y, W_i)(x_Y, y) \end{aligned} \quad (5.12)$$

or, with the alternative expression analogous to eqn. (5.9),

$$\begin{aligned} A(W_1, \dots, W_n; W_i)(x_1, \dots, x_n; y) \\ = \sum_{Y \in X} (-1)^{|Y|} T(W_{Y^c}, W_i)(x_{Y^c}, y) \bar{T}(W_Y)(x_Y). \end{aligned} \quad (5.13)$$

They have advanced support,

$$\begin{aligned} \text{supp } A(W_X; W_i)(x_X, y) \\ \subset \{(x_1, \dots, x_n, y) \in M^{n+1} : x_i \in (y + \bar{V}_+) \quad \forall x_i \in X\}. \end{aligned} \quad (5.14)$$

The interacting fields can also be defined in terms of advanced products instead of retarded products without changing the local field algebra if we define

$$(W_i)_{\text{int}}^{g\mathcal{L}}(f) = \int d^4 y f(y) \frac{\delta}{i\delta h(y)} S(g\mathcal{L} + hW_i) \Big|_{h=0} \times S(g\mathcal{L})^{-1}. \quad (5.15)$$

This would only result in a unitary transformation on $\tilde{\mathcal{F}}(\mathcal{O})$ with $S(g\mathcal{L})$ as the unitary operator.

Finally we define the causal product as

$$\begin{aligned} D(W_1, \dots, W_n; W_i)(x_1, \dots, x_n; y) \\ \stackrel{\text{def}}{=} R(W_1, \dots, W_n; W_i)(x_1, \dots, x_n; y) - A(W_1, \dots, W_n; W_i)(x_1, \dots, x_n; y) \end{aligned} \quad (5.16)$$

which has obviously causal support:

$$\begin{aligned} \text{supp } D(W_X; W_i)(x_X, y) \\ \subset \{(x_1, \dots, x_n, y) \in M^{n+1} : x_i \in (y + \overline{V}) \quad \forall x_i \in X\}. \end{aligned} \quad (5.17)$$

5.4. Conditions on the interaction Lagrangian. Up to now the Lagrangian density \mathcal{L} that defines the model via the S-matrix could have been an arbitrary polynomial in \mathcal{P} . There is a number of restrictions that such a polynomial must satisfy before it can define a reasonable physical model. In this section we will collect these restrictions.

At first, it must be Lorentz invariant:

$$(\text{Ad } U(p))T(\mathcal{L})(x) = T(\mathcal{L})(\Lambda^{-1}x) \quad \forall p = (0, \Lambda) \in \mathcal{P}_+^\uparrow. \quad (\text{C1})$$

The second condition it must satisfy is pseudo-unitarity:

$$(T(\mathcal{L}))^*(x) = T(\mathcal{L})(x). \quad (\text{C2})$$

Furthermore it must have vanishing ghost number,

$$s_c T(\mathcal{L})(x) = 0. \quad (\text{C3})$$

A Lagrangian with non vanishing ghost number would define a strange theory. The individual orders in perturbation theory of an interacting field would have a ghost number increasing (or decreasing) with the order. Such a theory would be super-renormalizable, provided it is power counting renormalizable, see below.

Since the S-matrix should also be BRS-invariant, one could also expect an equation like

$$s_0 T(\mathcal{L})(x) = 0 \quad (5.18)$$

to hold. Unfortunately it is in general — and specifically in QED and Yang-Mills-theory — impossible to find a Lagrangian for which eqn. (5.18) holds. So we must weaken the condition a little. Therefore we demand that there exist polynomials $\mathcal{L}_n^{\mu_1 \dots \mu_n}$ in \mathcal{P} , the so called $Q(n)$ -vertices, such that the following equations hold:

$$s_0 T(\mathcal{L}_n^{\mu_1, \dots, \mu_n})(x) = i \partial_\rho^x T(\mathcal{L}_{n+1}^{\mu_1, \dots, \mu_n, \rho})(x). \quad (\text{C4})$$

The $Q(n)$ -vertices must be totally antisymmetric in their Lorentz indices. The other index indicates the ghost number

$$g(\mathcal{L}_n) = n \mathcal{L}_n. \quad (5.19)$$

In (C4) there will be only a finite number of nontrivial equations, i.e. there exists an $m \in \mathbb{N}$ such that $\mathcal{L}_m = 0$. The $Q(n)$ -vertices have always the same canonical dimension as the original vertex \mathcal{L} , and they also contain the same number of generators. Therefore $\mathcal{L}_5 = 0$ for power counting renormalizable theories (see below) since \mathcal{L}_5 must have ghost number five and it is impossible to construct a polynomial with ghost number five and a canonical dimension not exceeding four. If the original vertex contains only three generators as it is usually the case then already $\mathcal{L}_4 = 0$. In Yang-Mills theory — with or without matter — even $\mathcal{L}_3 = 0$

and in QED $\mathcal{L}_2 = 0$. These results can be derived by explicit calculation.

The last condition on the Lagrangian we want to impose is power counting renormalizability. Perturbation theories can be divided into three groups according to the canonical dimension of their Lagrangian: Those with a canonical dimension less than the spacetime dimension are super renormalizable, that means the number of free normalization parameters decreases with the order and finally vanishes, so the theory is completely determined by a finite number of such parameters. Power counting renormalizable theories are those where the canonical dimension equals the space time dimension. For those theories there exists for all orders in perturbation theory a common upper bound for the number of free parameters in the extension. Non renormalizable theories have Lagrangians whose canonical dimension exceeds the spacetime dimension, and this leads to a number of free normalization parameters that may increase with the order. Although the predictive power of such theories — perturbative gravitation is an example of those — is rather poor, it is nevertheless possible to deal with them in the framework of causal perturbation theory.

For our considerations non renormalizable Lagrangians play no role and therefore we exclude them explicitly. As we always work in four spacetime dimensions, the condition for renormalizability reads

$$\text{deg } \mathcal{L} \leq 4, \tag{C5}$$

where deg means the canonical dimension.

6. THE INTERACTING THEORY

In this chapter we come back to the program for the construction of interacting gauge theories outlined in chapter (2). We formulated at the end of section (2.3) four requirements for an interacting gauge theory. With the construction of local interacting field theories in the last chapter and the normalization conditions in chapter (4) we are now able to determine under which conditions these requirements can be accomplished. The first condition — the condition that suitable ghost and BRS charges can be found in the free model — must be verified for the individual model. This has been done for the free models underlying QED and Yang-Mills theory in section (3.5). In this chapter we will see that the other three conditions hold if all normalization conditions (N1) - (N6) are satisfied and if the conditions (C1) - (C5) are valid for the Lagrangian \mathcal{L} which defines the model. We assume throughout this chapter that these preconditions hold.

In the first section we collect a number of properties all interacting fields share from their very definition. Among them are e.g. covariance and locality. In addition we derive a relation between the interacting field operators for the higher generators and those for the basic generators.

In the second section we formulate field equations for the interacting field operators. These equations are determined by normalization condition (N4).

In the third section we come to interacting operators that are of particular importance in gauge theories. In this section we define the interacting ghost current, the interacting ghost charge and the ghost number of interacting fields. We prove that the interacting ghost current is conserved and that the higher order contributions of the ghost charge vanish. As a consequence every interacting field has the same ghost number as the corresponding free field.

In the fourth section we define the most essential operators in an interacting gauge theory: the interacting BRS current, the interacting BRS charge and the interacting BRS transformation. We find that the interacting BRS current is conserved only where the test function g that defines the coupling is constant. The BRS charge is constructed only for spacetimes that are compactified in spacelike directions. Otherwise its definition would not be well posed. We prove also that with our definitions the BRS algebra holds. This means in particular that the interacting BRS charge is nilpotent.

In the last section we examine the relation between the quantum field theory defined above and its corresponding classical theory and formulate a correspondence law for these theories.

6.1. General properties of interacting fields. We begin our considerations with

C-numbers: From the definition of the retarded products, eqn. (5.8), we can find that they vanish if at least one of their arguments is a multiple of the identity — provided the total number of arguments is at least two, see [DF99]. This implies immediately for interacting fields that are generated by \mathbb{C} -numbers that they possess no higher order terms:

$$(\alpha \cdot \mathbb{1})_{\text{int}}^{g\mathcal{L}}(x) = \alpha \cdot \mathbb{1}, \quad \alpha \in \mathbb{C}. \quad (6.1)$$

Lorentz covariance: The fact that the Lagrangian is a Lorentz scalar implies, together with condition (N1), the Lorentz transformation properties of the interacting

field operators:

$$(\text{Ad } U(p))(W_i)_{\text{int}}^{g\mathcal{L}}(x) = (\mathcal{R}_\Lambda(W_i))_{\text{int}}^{g^p\mathcal{L}}(x-a), \quad \forall p = (a, \Lambda) \in \mathfrak{P}_+^\dagger \quad (6.2)$$

where \mathcal{R} is the representation of the Lorentz group (or its covering group) defined in section (3.1) and $g^p = g(\Lambda^{-1}x - a)$.

Pseudo-hermiticity: Due to the conditions **(C2)** and **(N2)** the Krein adjoint of the interacting fields is given by

$$\left((W_i)_{\text{int}}^{g\mathcal{L}}(x)\right)^* = \left(W_i^*\right)_{\text{int}}^{g\mathcal{L}}(x) \quad \forall W_i \in \mathcal{P}. \quad (6.3)$$

The *-involution on the right hand side is the one introduced in section (3.1).

Locality: A very important property of interacting fields is their locality. This means that two interacting field operators (anti-) commute with each other if they are localized in spacelike separated regions. This can immediately be derived from eqn. (5.11):

$$\left[(W^1)_{\text{int}}^{g\mathcal{L}}(x), (W^2)_{\text{int}}^{g\mathcal{L}}(y)\right]_{\mp} = 0 \quad \text{if } x \not\sim y. \quad (6.4)$$

Primary interacting fields: Due to normalization condition **(N4)** the interacting fields for the higher generators may be expressed by those for the basic generators as:

$$\begin{aligned} \left((\varphi_i)^{(n, \nu_1 \dots \nu_n)}\right)_{\text{int}}^{g\mathcal{L}}(x) &= \partial_x^{\nu_1} \dots \partial_x^{\nu_n} \left((\varphi_i)^{(0)}\right)_{\text{int}}^{g\mathcal{L}}(x) \\ &+ C_{\varphi_i, n} g(x) \left(\frac{\partial \mathcal{L}}{\partial \tilde{\varphi}_i^{(n, \nu_1 \dots \nu_n)}}\right)_{\text{int}}^{g\mathcal{L}}(x), \end{aligned} \quad (6.5)$$

where $\tilde{\varphi}_i$ is the field conjugated to φ_i .

6.2. The interacting field equations. Now we state field equations for the interacting field theory. They are again already determined by condition **(N4)** and read

$$\sum_j D_{ij}^x (\varphi_j)_{\text{int}}^{g\mathcal{L}}(x) = -g(x) \left(\frac{\partial \mathcal{L}}{\partial \varphi_i}\right)_{\text{int}}^{g\mathcal{L}}(x). \quad (6.6)$$

Inserting here the definition of D^x — eqn. (3.86) and the following ones — we find that this implies in particular

$$K^{\varphi_i, x} \left((\varphi_i)^{(0)}\right)_{\text{int}}^{g\mathcal{L}}(x) = - \sum_{n=0}^{\infty} (-1)^n \partial_x^{\nu_1} \dots \partial_x^{\nu_n} \left(g(x) \left(\frac{\partial \mathcal{L}}{\partial (\tilde{\varphi}_i)^{(n, \nu_1 \dots \nu_n)}}\right)_{\text{int}}^{g\mathcal{L}}(x)\right), \quad (6.7)$$

where $K^{\varphi_i, x}$ was defined in eqn. (3.88). These are exactly the field equations that are derived as the Euler-Lagrange equations for a classical field theory with a Lagrangian $\mathcal{L}_0 + \mathcal{L}$, where \mathcal{L} is the interaction Lagrangian and \mathcal{L}_0 is the free Lagrangian that implies the free field equations

$$K^{\varphi_i, x} \varphi_i(x) = 0, \quad \varphi_i(x) \text{ a classical field.} \quad (6.8)$$

But there is one important difference between the field equations in the classical theory and those in the quantum theory. While the classical field equations govern the dynamics of the system, this is not true for the quantum field equations. The reason is that the classical theory has fewer independent variables. The field

equations determine the time evolution of the basic fields on the left hand side. Therefore the time evolution of the entire classical theory is determined by the field equations, since all variables are basic fields or products thereof. This is not true in the quantum theory, because the interacting fields for composed elements in the algebra \mathcal{P} are not products of those for the generators¹⁹. Therefore the time evolution of the interacting fields for composed elements of \mathcal{P} is left open by the equations above.

The quantum field equations are completely independent of the normalization constants $C_{\varphi_i, k}$ in eqn. (3.78). They are also independent of the normalization of time ordered products, provided condition (N4) applies.

6.3. The interacting ghost current and the ghost charge. The interacting ghost current is defined as the interacting field operator that is generated by the free ghost current k^μ , see section (3.5):

$$\tilde{k}^\mu(x) \stackrel{\text{def}}{=} (k^\mu)_{\text{int}}^{g\mathcal{L}}(x). \quad (6.9)$$

This current is conserved as is easily derived by means of (N5) and (C3):

$$\partial_\mu^x \tilde{k}^\mu(x) = 0. \quad (6.10)$$

From eqn. (6.3) and the fact that the free ghost current is anti-pseudo-hermitian we find that the interacting ghost current is anti-pseudo-hermitian, too:

$$\left(\tilde{k}^\mu(x)\right)^* = -\tilde{k}^\mu(x). \quad (6.11)$$

The interacting ghost charge is defined as

$$\tilde{Q}_c \stackrel{\text{def}}{=} \lim_{\lambda \searrow 0} \int d^4y h_\lambda(y) \tilde{k}^0(y), \quad (6.12)$$

where $h_\lambda(x^0, \mathbf{x}) = \lambda h^t(\lambda x^0) b(\lambda \mathbf{x})$, see eqn. (3.103). Here the coordinate frame is chosen such that the origin 0 is in the domain \mathcal{O} where the fields are localized. We restrict the admissible spatial test functions b : At first the temporal test function h^t is selected such that $0 \in \text{supp } h^t$ and the following equation holds:

$$(\text{supp}(\partial g) \cap [\mathcal{O} + \overline{V}_+]) \supseteq (\text{supp } h^t \times \mathbb{R}^3) \supseteq (\text{supp}(\partial g) \cap [\mathcal{O} + \overline{V}_-]). \quad (6.13)$$

Then only test functions b are admitted with the following properties: $b(\mathbf{x}) = 1$ for all $\mathbf{x} \in \mathbb{R}^3$ for which an $x^0 \in \text{supp } h^t$ exists such that

$$(x^0, \mathbf{x}) \in (\text{supp } g + \overline{V}_+). \quad (6.14)$$

The question arises whether the limit in the definition of \tilde{Q}_c exists. We will show that this is indeed true.

The zeroth order of the interacting ghost current is simply the free ghost current. For the free current we know already that the limit exists, so we confine our attention to the higher orders.

We will prove that the higher orders of the ghost charge do not depend on λ . For this purpose we calculate for the n^{th} order of the ghost charge, $n \geq 1$ and $\lambda \leq 1$:

$$Q_{c, \lambda}^n - Q_{c, 1}^n = \int d^4x \left(h_\lambda(x) - h_1(x) \right) \tilde{k}^{0, n}(x). \quad (6.15)$$

¹⁹A product of distributional field operators is not defined a priori. It can be examined in the framework of operator product expansions [Wil69, Wil71, Zim73], but we will not discuss this here.

Here $\tilde{k}^{\mu,n}$ is the n^{th} order of the ghost current. We have for all $n \geq 1$ that $\text{supp } \tilde{k}^{\mu,n} \subset (\text{supp } g + \bar{V}_+)$ due to the support properties of the retarded products.

With our conventions for the test functions we can substitute in eqn. (6.15) on the right hand side $h^t(x^0)b(\lambda x)$ for $h_1(x) = h^t(x^0)b(x)$ because $h^t(x^0)(b(\lambda x) - b(x))$ vanishes on the support of $\tilde{k}^{\mu,n}$, $n \geq 1$. Then eqn. (6.15) becomes

$$Q_{c,\lambda}^n - Q_{c,1}^n = \int d^4x \left(\lambda h^t(\lambda x^0) - h^t(x^0) \right) b(\lambda \mathbf{x}) \tilde{k}^{0,n}(x). \quad (6.16)$$

There exists a test function $H_\lambda \in \mathcal{D}(\mathbb{R})$ such that

$$\partial_0^x H_\lambda(x^0) = (\lambda h^t(\lambda x^0) - h^t(x^0)). \quad (6.17)$$

Inserting this into (6.15) we get

$$\begin{aligned} Q_{c,\lambda}^n - Q_{c,1}^n &= \int d^4x \left(\partial_0^x H_\lambda(x^0) \right) b(\lambda \mathbf{x}) \tilde{k}^{0,n}(x) \\ &= - \int d^4x H_\lambda(x^0) \left(\partial_i^x b(\lambda \mathbf{x}) \right) \tilde{k}^{i,n}(x), \end{aligned} \quad (6.18)$$

where we have partially integrated and used the fact that \tilde{k}^μ is conserved. By construction we have

$$\text{supp} \left(H_\lambda(x^0) \left(\partial_i^x b(\lambda \mathbf{x}) \right) \right) \cap (\text{supp } g + \bar{V}_+) = \emptyset. \quad (6.19)$$

Comparing this with the support of $\tilde{k}^{\mu,n}$, we see that the integral vanishes. Therefore the higher orders of \tilde{Q}_c do not depend on λ . Even more, because of current conservation, eqn. (6.10), one can choose h^t such that the support of h_1 is entirely in the past of $\text{supp } g$. Then the higher order terms vanish due to the support properties of the retarded products, so the interacting ghost charge coincides with the free ghost charge or, strictly speaking since \tilde{Q}_c is a formal power series,

$$\tilde{Q}_c = (Q_c, 0, 0, \dots). \quad (6.20)$$

Since the ghost current is anti-pseudo-hermitian, the ghost charge is it, too:

$$\tilde{Q}_c^* = -\tilde{Q}_c. \quad (6.21)$$

The interacting ghost number of a localized field operator is measured by the following derivation:

$$\tilde{s}_c \left((W_i)_{\text{int}}^{g\mathcal{L}}(x) \right) \stackrel{\text{def}}{=} \left[\tilde{Q}_c, (W_i)_{\text{int}}^{g\mathcal{L}}(x) \right]_-. \quad (6.22)$$

As the interacting ghost charge coincides with the free one, we have

$$\tilde{s}_c \left((W_i)_{\text{int}}^{g\mathcal{L}}(x) \right) = s_c \left((W_i)_{\text{int}}^{g\mathcal{L}}(x) \right). \quad (6.23)$$

This implies immediately, due to **(N5)** and **(C3)**, that the interacting field operators have the same ghost number as the corresponding free fields:

$$\tilde{s}_c \left((W_i)_{\text{int}}^{g\mathcal{L}}(x) \right) = g(W_i) (W_i)_{\text{int}}^{g\mathcal{L}}(x), \quad g(W_i) \in \mathbb{Z}. \quad (6.24)$$

6.4. The interacting BRS current, BRS charge and BRS transformation.

The natural choice for the BRS current,

$$\tilde{\mathcal{J}}_B^\mu(x) = (j_B^\mu)^{g\mathcal{L}}(x), \quad (6.25)$$

is not conserved in general, so this cannot be the correct interacting BRS current. The situation is even worse: Explicit calculations in first order QED and Yang-Mills theory shows that there exists no normalization of the time ordered products such that this current is conserved even in first order, irrespective of our normalization conditions. The best one can achieve is that the current is conserved where the coupling is constant, and even this seemingly liberal condition fixes the normalization in first order uniquely.

A direct calculation reveals that this normalization is not compatible with the normalization conditions (N3) and (N4). But there is an expression for the interacting BRS current that is compatible with the normalization conditions in first order and that is conserved in the sense above, not only for Yang-Mills theories but for every theory. Adopting this expression as the definition of the interacting BRS current we have

$$\tilde{\mathcal{J}}_B^\mu(x) \stackrel{\text{def}}{=} (j_B^\mu)^{g\mathcal{L}}(x) - g(x)(\mathcal{M}_1^\mu)^{g\mathcal{L}}(x), \quad (6.26)$$

where \mathcal{M}_1^μ is the $R(1)$ -vertex, see condition (C4). Normalization condition (N6) implies that this current is indeed conserved where the coupling g is constant:

$$\partial_\mu^x \tilde{\mathcal{J}}_B^\mu(x) = (\partial_\nu g)(x)(\mathcal{L}_1^\nu)^{g\mathcal{L}}(x). \quad (6.27)$$

The fact that the interacting BRS current is not everywhere conserved is a severe drawback, since it complicates the definition of the BRS charge, see below. So the question arises whether a more clever choice for the BRS current could have yielded one that is everywhere conserved. But this turns out to be impossible in general. Concretely, in QED as well as in Yang-Mills theory the explicit calculation shows already in first order that no such choice exists. So in general this result cannot be improved.

By the same reasoning as for eqn. (6.11) one derives that $\tilde{\mathcal{J}}_B^\mu$ is pseudo-hermitian

$$(\tilde{\mathcal{J}}^\mu(x))^* = \tilde{\mathcal{J}}^\mu(x). \quad (6.28)$$

Now we come to the definition of the BRS charge. As already mentioned this definition is more difficult than that of the ghost charge was, since the BRS current is not everywhere conserved. The problem can be seen as follows: The natural choice for the BRS charge would be

$$\tilde{Q}_B = \lim_{\lambda \searrow 0} \int d^4y h_\lambda(y) \tilde{\mathcal{J}}_B^0(y), \quad (6.29)$$

with h_λ like above. Unfortunately this expression would depend on the choice of h_λ , unlike for \tilde{Q}_c , and the higher orders would depend on λ , both because the BRS current is not conserved. If the higher orders depend on λ the limit is no longer under control.

In order to make \tilde{Q}_B independent of h_λ , the support of h_λ must be for every λ in a region where g is constant. This would mean that g is everywhere constant, i.e. the adiabatic limit must be performed, and this limit does not exist in general.

Another possibility would be not to perform the limit and choose e.g. h_1 as a test function in the definition of the BRS charge. In this case the BRS charge would clearly become well defined, but it would also be a local operator, and such

an operator could not annihilate states with finite energy due to the theorem of Reeh and Schlieder. It is unlikely that the cohomology defined with it has good properties, and therefore we exclude this possibility.

The way out of this seemingly pitfall was found by Dütsch and Fredenhagen [DF99]. In order to allow functions that are constant in spacelike directions as test functions, they embed the double cone \mathcal{O} isometrically into the cylinder $\mathbb{R} \times C_L$ with \mathbb{R} the time axis and C_L a cube of length L sufficiently big to contain \mathcal{O} . This spatial compactification does not change the properties of the local algebra $\tilde{\mathcal{F}}(\mathcal{O})$. This is why the quantization of free fields in a box, mentioned in chapter (3), is important for us. For the details of the construction we refer to [DF99].

In the compactified space h and g can be chosen to be test functions such that g is constant on $\text{supp } h$ with the same value as on \mathcal{O} .

With these test functions we are able to give a definition of the interacting BRS current in a spatially compactified spacetime. At first, we choose the test function h to be

$$h(x) = h^t(x_0), \quad h^t \text{ like in (3.103)}, \quad \implies h \in \mathcal{D}(\mathbb{R} \times C_L), \quad (6.30)$$

and the coupling g such that

$$g|_{\text{supp } h} = g|_{\mathcal{O}} = \text{constant}, \quad g \in \mathcal{D}(\mathbb{R} \times C_L). \quad (6.31)$$

With g and h now both being a test function — on $\mathbb{R} \times C_L$ — the BRS charge can be defined as

$$\tilde{Q}_B \stackrel{\text{def}}{=} \int_{\mathbb{R} \times C_L} d^4y h(y) \tilde{j}_B^0(y). \quad (6.32)$$

It is easy to see by an analogous reasoning as for the ghost charge that this BRS charge is independent of h .

The zeroth order of this BRS charge agrees with the free BRS charge in the $\mathbb{R} \times C_L$ spacetime, $\tilde{Q}_{B,0} = Q_B$, if h_λ is replaced there by h . Of course the limit is then not performed because it would be void. Unlike the interacting ghost charge the interacting BRS charge has also non vanishing higher order contributions. The reasoning which showed that the higher contributions of \tilde{Q}_c vanish cannot be applied here, since $\text{supp } h$ may not be (not even partly) in the past of $\text{supp } g$ from its very definition.

Like the BRS current the BRS charge is pseudo-hermitian:

$$\tilde{Q}_B^* = \tilde{Q}_B. \quad (6.33)$$

Since $s_c Q_B = Q_B$ in the free theory, we find with eqn. (4.32)

$$\left[\tilde{Q}_c, \tilde{Q}_B \right]_- = \tilde{s}_c \left(\tilde{Q}_B \right) = \tilde{Q}_B. \quad (6.34)$$

So the first part of the BRS algebra holds. The most important property of the BRS charge is its nilpotency, the second part of the BRS algebra. This will be proven next.

To this end we write at first \tilde{Q}_B in a different form that is more adequate for the proof. We use for the interacting field $(j^\mu)_{\text{int}}^{g\mathcal{L}}(x)$ in the definition of \tilde{j}^μ the equation (B.25) from the appendix and the identity $\tilde{Q}_c = Q_c$, eqn. (6.20). With it

the interacting BRS charge can be written as

$$\begin{aligned} \tilde{Q}_B = Q_B + \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^4 y d^4 z d^4 x_1 \cdots d^4 x_n h(y) (\partial_\nu g)(z) \times \\ \times g(x_1) \cdots g(x_n) R\left(\mathcal{L}, \dots, \mathcal{L}, \mathcal{L}_1^\nu; k^0\right) (x_1, \dots, x_n, z; y). \end{aligned} \quad (6.35)$$

The first term in the sum on the right hand side is the free BRS charge. Since its properties are already known, we confine our attention to the higher order terms. From the form of these terms one can see immediately that the interacting BRS charge becomes the free one in the adiabatic limit, if this limit exists. We are here particularly interested in theories where the adiabatic limit does not necessarily exist.

We introduce a test function $H \in \mathcal{D}(\mathbb{R} \times C_L)$ with the property

$$\partial_\mu^y H(y) = -\delta_\mu^0 h(y), \quad H \in \mathcal{D}(\mathbb{R} \times C_L), \quad (6.36)$$

such that $H(x) = 1$ for all x in the past of $\text{supp } g$ and $H(x) = 0$ for all x in the future of $\text{supp } g$. Inserting this into the expression above, we find by partial integration and with the help of eqn. (N5) the following alternative formulation for the n^{th} order of \tilde{Q}_B , $n \geq 1$:

$$\begin{aligned} \tilde{Q}_B^{(n)} = \frac{i^{n-1}}{(n-1)!} \int d^4 z d^4 x_1 \cdots d^4 x_{n-1} H(z) (\partial_\nu g)(z) \times \\ \times g(x_1) \cdots g(x_{n-1}) R\left(\mathcal{L}, \dots, \mathcal{L}; \mathcal{L}_1^\nu\right) (x_1, \dots, x_{n-1}; z). \end{aligned} \quad (6.37)$$

The n^{th} order of $(\tilde{Q}_B)^2$ decomposes according to

$$\left((\tilde{Q}_B)^2\right)^{(n)} = \sum_{k=0}^n (\tilde{Q}_B)^{(k)} (\tilde{Q}_B)^{(n-k)} = s_0 \left((\tilde{Q}_B)^{(n)}\right) + \sum_{k=1}^{n-1} (\tilde{Q}_B)^{(k)} (\tilde{Q}_B)^{(n-k)}. \quad (6.38)$$

At first we will calculate $s_0 \left((\tilde{Q}_B)^{(n)}\right)$. With the help of the generalized operator gauge invariance, eqn. (4.35), with $i_1 = 1$ and $i_k = 0$ otherwise, we get

$$\begin{aligned} s_0(\tilde{Q}_B)^{(n)} = \\ = \frac{i^{n-2}}{(n-2)!} \int d^4 x_1 \cdots d^4 x_{n-2} d^4 y d^4 z g(x_1) \cdots g(x_{n-2}) \\ \times (\partial_\rho g)(y) H(y) (\partial_\mu g)(z) H(z) R\left(\mathcal{L}, \dots, \mathcal{L}, \mathcal{L}_1^\rho; \mathcal{L}_1^\mu\right) (x_1, \dots, x_{n-2}, y; z) \\ + \frac{i^n}{(n-1)!} \int d^4 x_1 \cdots d^4 x_{n-1} d^4 z g(x_1) \cdots g(x_{n-1}) \\ \times \left(\partial_\rho^z [(\partial_\mu g)(z) H(z)]\right) R\left(\mathcal{L}, \dots, \mathcal{L}; \mathcal{L}_2^{\mu\rho}\right) (x_1, \dots, x_{n-1}; z). \end{aligned} \quad (6.39)$$

Here an additional factor $H(y)$ has been inserted in the first integral. This factor does not change the result due to the retarded support of the distribution.

Let us at first consider the second integral on the right hand side. Calculating the derivative of the square bracket in the last line, we get $(\partial_\mu \partial_\rho g)(z) H(z) + (\partial_\mu g)(z) (\partial_\rho H)(z)$. The second term vanishes since the supports of ∂g and ∂H

are disjoint. The first term is symmetric in μ and ρ while the retarded product is antisymmetric in these indices, due to the antisymmetry of the $Q(2)$ -vertex. Therefore the entire second integral vanishes.

Now we come to the first integral. Here the test functions are also symmetric under permutation of (z, μ) and (y, ρ) . If we look at the definition of the retarded products, eqn. (5.8), we see that there are terms where both \mathcal{L}_1^μ and \mathcal{L}_1^ρ appear as arguments in the same time ordered product or antichronological product. These contributions vanish, because the distributions are antisymmetric in (z, μ) and (y, ρ) due to graded symmetry (**P2**). The only contributions that remain lead to our final expression for $s_0(\tilde{Q}_B)^{(n)}$:

$$\begin{aligned} s_0(\tilde{Q}_B)^{(n)} &= \\ &= -\frac{i^{n-2}}{(n-2)!} \int d^4x_1 \cdots d^4x_{n-2} d^4y d^4z g(x_1) \cdots g(x_{n-2}) (\partial_\rho g)(y) H(y) \\ &\quad \times (\partial_\mu g)(z) H(z) \sum_{Y \subset X} (-1)^{|Y|} \bar{T}(\mathcal{L}, \dots, \mathcal{L}, \mathcal{L}_1^\rho)(x_Y, y) T(\mathcal{L}, \dots, \mathcal{L}, \mathcal{L}_1^\mu)(x_{Y^c}, z) \end{aligned} \quad (6.40)$$

with $X = \{x_1, \dots, x_{n-2}\}$.

To calculate $\sum_{k=1}^{n-1} (\tilde{Q}_B)^{(k)} (\tilde{Q}_B)^{(n-k)}$ we make use of the two ways to express R -products in terms of T - and \bar{T} -products, that means we use eqn. (6.37) for the individual orders of the BRS charge, inserting eqn. (5.9) for the retarded products on the left hand side and eqn. (5.8) for those on the right hand side. Then we get after a little combinatorial analysis

$$\begin{aligned} \sum_{k=1}^{n-1} (\tilde{Q}_B)^{(k)} (\tilde{Q}_B)^{(n-k)} &= \\ &= \frac{i^{n-2}}{(n-2)!} \int d^4x_1 \cdots d^4x_{n-2} d^4y d^4z g(x_1) \cdots g(x_{n-2}) (\partial_\rho g)(y) H(y) (\partial_\mu g)(z) \\ &\quad \times H(z) \left[\sum_{Y, Z, U, V} (-1)^{|Z|+|V|} \left(\bar{T}(\mathcal{L}, \dots, \mathcal{L}, \mathcal{L}_1^\mu)(x_Z, z) T(\mathcal{L}, \dots, \mathcal{L})(x_Y) \right) \right. \\ &\quad \left. \times \left(\bar{T}(\mathcal{L}, \dots, \mathcal{L})(x_V) T(\mathcal{L}, \dots, \mathcal{L}, \mathcal{L}_1^\rho)(x_U, y) \right) \right], \end{aligned} \quad (6.41)$$

where the sum in the square brackets runs over all disjoint partitions of X into four subsets U, V, Y, Z . These subsets may be empty. This set of partitions can be divided into two subsets, namely the set of those partitions where Y and V are empty and its complement. This complement can in turn be divided in subsets with U and Z fixed, yielding terms proportional to

$$\sum_{W \subset X \setminus U \setminus Z} (-1)^{|W|} T(\mathcal{L}, \dots, \mathcal{L})(x_W) \bar{T}(\mathcal{L}, \dots, \mathcal{L})(x_{X \setminus U \setminus Z \setminus W}). \quad (6.42)$$

This expression vanishes due to eqn. (4.21) because $X \setminus U \setminus Z \neq \emptyset$ according to our assumption. So there remains only a contribution from the partitions with

$Y = V = \emptyset$, and since $T_0 = \bar{T}_0 = \mathbb{1}$, there remains only

$$\begin{aligned} & \sum_{k=1}^{n-1} (\tilde{Q}_B)^{(k)} (\tilde{Q}_B)^{(n-k)} = \\ & = \frac{i^{n-2}}{(n-2)!} \int d^4 x_1 \cdots d^4 x_{n-2} d^4 y d^4 z g(x_1) \cdots g(x_{n-2}) (\partial_\rho g)(y) H(y) \\ & \quad \times (\partial_\mu g)(z) H(z) \sum_{Y \subset X} (-1)^{|Y|} \bar{T}(\mathcal{L}, \dots, \mathcal{L}, \mathcal{L}_1^\rho)(x_Y, y) T(\mathcal{L}, \dots, \mathcal{L}, \mathcal{L}_1^\mu)(x_{Y^c}, z). \end{aligned} \quad (6.43)$$

Obviously this is just the negative of eqn. (6.40), yielding

$$\left((\tilde{Q}_B)^2 \right)^{(n)} = s_0 \left((\tilde{Q}_B)^{(n)} \right) + \sum_{k=1}^{n-1} (\tilde{Q}_B)^{(k)} (\tilde{Q}_B)^{(n-k)} \stackrel{!}{=} 0. \quad (6.44)$$

Reviewing our preconditions, we have proven that with our definition the BRS charge is nilpotent — and therefore the complete BRS algebra holds —, provided our normalization condition **(N6)** is valid.

At the end of this section we come to the interacting BRS transformation \tilde{s} . It could be defined as

$$\tilde{s} \left((W)_{\text{int}}^{g\mathcal{L}}(x) \right) = \left[\tilde{Q}_B, (W)_{\text{int}}^{g\mathcal{L}}(x) \right]_{\mp}. \quad (6.45)$$

But it turns out to be more clever to permute commutation and integration, and we define

$$\tilde{s} \left((W)_{\text{int}}^{g\mathcal{L}}(x) \right) \stackrel{\text{def}}{=} \int_{\mathbb{R} \times C_L} d^4 y h(y) \left[\tilde{j}_B^0(y), (W)_{\text{int}}^{g\mathcal{L}}(x) \right]_{\mp} \quad (6.46)$$

with h and g as in the definition of \tilde{Q}_B . The advantage of this definition is that it remains well defined for \tilde{s} acting on local fields in $\tilde{\mathcal{F}}(\mathcal{O})$ even if the spacetime is not compactified and h has compact support only in timelike directions being constant in spacelike directions. This is well defined because locality, eqn. (6.4), holds — both \tilde{j}_B^0 and $(W)_{\text{int}}^{g\mathcal{L}}(f)$ are local fields. Therefore the commutator has causal support, so the integrand vanishes in the causal complement of \mathcal{O} .

$$\tilde{s} \left((W)_{\text{int}}^{g\mathcal{L}}(f) \right) \stackrel{\text{def}}{=} \int d^4 y h(y) \left[\tilde{j}_B^0(y), (W)_{\text{int}}^{g\mathcal{L}}(f) \right]_{\mp} \quad (6.47)$$

is a well defined expression in Minkowski space, if h is chosen such that

$$\begin{aligned} h(x) &= h^t(x_0), \quad h^t \in \mathcal{D}(\mathbb{R}) \text{ as in eqn. (3.103),} \\ g &\text{ is constant on } (\mathcal{O} + \bar{V}_-) \cap (\text{supp } h + \bar{V}_+). \end{aligned} \quad (6.48)$$

This expression is independent of h . The BRS transformation is nilpotent. This can be seen by direct computation — the calculation is then completely analogous to that for $(\tilde{Q}_B)^2 = 0$ in the compactified spacetime. A different way to prove that \tilde{s} is nilpotent is to consider \tilde{s} in a compactified spacetime — where $\tilde{s}^2 = 0$ follows directly from $(\tilde{Q}_B)^2 = 0$. Then let the compactification length L tend to infinity. The resultant space will be the Minkowski space and $\tilde{s}^2 = 0$ still holds since the algebra does not depend on the compactification length. Therefore

$$\tilde{s}^2 A = 0 \quad \forall A \in \tilde{\mathcal{F}}(\mathcal{O}). \quad (6.49)$$

It is important to note that this reasoning holds only for local operators. In particular the argument of Nakanishi and Ojima [NO90] that a nilpotent BRS transformation defines a nilpotent BRS charge can not be applied here. Their argument is as follows:

$$\tilde{Q}_B \stackrel{\text{def}}{=} -\tilde{s}\tilde{Q}_c \quad \text{and} \quad 0 = \tilde{s}^2(\tilde{Q}_c) = -\tilde{s}(\tilde{Q}_B) = -2\tilde{Q}_B^2, \quad (6.50)$$

but since \tilde{Q}_c is not a local operator it is not in the domain of \tilde{s} in the framework of ordinary spacetime.

So we arrive at the following result: For all investigations concerning the state space it is necessary to compactify spacetime, since we need the BRS charge to define the physical state space, and this is only defined in a compactified spacetime. But for investigations concerning only the algebra of local observables there is no need for a compactification because the definition of observables requires only the BRS transformation, not the BRS charge, and the former can also be defined in an ordinary spacetime.

Summarizing the results of this and the preceding chapter we see that all the preconditions that we postulated at the end of section (2.3) are satisfied. The only restriction is that the BRS current is conserved only locally, but this is sufficient for the construction of the local interacting gauge theory.

This result was derived under the assumption that the normalization conditions (N1) - (N6) and the conditions on the Lagrangian are satisfied. We proved in chapter (4) that the first five normalization conditions have simultaneous solutions. So the essential point is whether condition (N6) can be satisfied for a model. If this is the case, the construction of the physical state space (in the spatially compactified spacetime) and of the local observable algebra can be performed.

6.5. The correspondence between quantum and classical theory. We have seen in section (4.4) that in our approach the propagators for the higher generators are different from the corresponding propagators for the derivated fields in other renormalization procedures. The propagators determine the tree diagrams, and these in turn are known to determine the classical limit of the theory. Therefore the question arises whether the classical limit of our theory is different from what one would expect from other approaches. We will see that this is indeed the case. The classical fields are functions on a manifold, in this case the Minkowski space. Unlike the distributional field operators they may be multiplied at the same spacetime point. We take advantage of this property and define a representation C of the algebra \mathcal{P} by classical fields. Unlike the representation T of \mathcal{P} in section (3.3) this is not only a linear representation but also an algebra homomorphism. We define

$$C : \mathcal{P} \rightarrow C^\infty(M), \quad C(a \cdot A) = a \cdot C(A) \quad \forall a \in \mathbb{C}, A \in \mathcal{P},$$

$$C \left(\prod_i \varphi_i \right) (x) = \prod_i C(\varphi_i)(x), \quad \varphi_i \in \mathcal{G}. \quad (6.51)$$

The representatives of the basic generators are the basic classical fields, i.e. we suppose that there exists for each $\varphi_i \in \mathcal{G}_b$ a classical field $\varphi_i^{\text{cl}}(x)$ such that

$$C(\varphi_i)(x) = \varphi_i^{\text{cl}}(x). \quad (6.52)$$

The question arises how the higher generators may be represented. The first attempt is to define their representatives as the derivatives of those for the basic

generators, e.g.

$$C\left((\varphi_i)^{(1,\mu)}\right)(x) = \partial_x^\mu C\left((\varphi_i)^{(0)}\right)(x). \quad (6.53)$$

But this definition is not consistent. This can be seen by comparing this equation with eqn. (6.5),

$$\begin{aligned} \left((\varphi_i)^{(n,\nu_1\dots\nu_n)}\right)_{\text{int}}^{g\mathcal{L}}(x) &= \partial_x^{\nu_1} \dots \partial_x^{\nu_n} \left((\varphi_i)^{(0)}\right)_{\text{int}}^{g\mathcal{L}}(x) \\ &+ C_{\varphi_i,n} g(x) \left(\frac{\partial \mathcal{L}}{\partial \tilde{\varphi}_i^{(n,\nu_1\dots\nu_n)}}\right)_{\text{int}}^{g\mathcal{L}}(x). \end{aligned} \quad (6.54)$$

If we adopted the definition above, the left hand side and the right hand side of this equation would be equal on the quantum level, but they would have different classical limits, and this cannot be true. We see that the correct prescription for the classical limit of the higher generators is

$$\begin{aligned} C\left((\varphi_i)^{(n,\nu_1\dots\nu_n)}\right)(x) &= \partial_x^{\nu_1} \dots \partial_x^{\nu_n} C\left((\varphi_i)^{(0)}\right)(x) \\ &+ g C_{\varphi_i,n} C\left(\frac{\partial \mathcal{L}}{\partial \tilde{\varphi}_i^{(n,\nu_1\dots\nu_n)}}\right)(x). \end{aligned} \quad (6.55)$$

Here we have set the coupling g constant, since in a classical theory there is no need for the interaction to be switched off. $\tilde{\varphi}_i$ is the generator of the field conjugate to φ_i .

The fields that correspond to the higher generators are labelled by the normalization constants $C_{\varphi_i,n}$. This is what we expected when we pointed out the importance of the propagators for the classical limit, because these propagators are also labelled by the normalization constants.

With the representation C now defined we can formulate the correspondence law. It states that the distributional interacting field operators become products of classical fields in the classical limit according to

$$\begin{aligned} (W)_{\text{int}}^{g\mathcal{L}}(x) &\rightarrow C(W)(x) \quad \forall W \in \mathcal{P}. \\ g(x) &\rightarrow g = \text{constant}. \end{aligned} \quad (6.56)$$

7. TWO PARTICULAR THEORIES

In this chapter we will examine the consequences of our general results derived in the preceding chapter for two well known models: Quantum electrodynamics and Yang-Mills theory.

7.1. Quantum electrodynamics. The fields involved in QED are vector bosons A_μ — the photons —, ghosts and anti-ghosts u, \tilde{u} and charged spinors $\psi, \bar{\psi}$ — the electrons and positrons.

For QED there exists a way to determine the physical state vector space without the BRS formalism — the Gupta-Bleuler procedure. Furthermore the ghosts do not couple to the other fields. Therefore it is not necessary to include the ghosts in the model. Nevertheless we do so because we investigate QED also as a preparation for Yang-Mills theory where the ghosts are indispensable.

The corresponding free theory for QED has been treated in section (3.5).

Therefore we start directly with the interaction. The interaction Lagrangian for QED reads

$$\mathcal{L}_{QED} = A_\mu j_{\text{el}}^\mu \in \mathcal{P}. \quad (7.1)$$

Here A_μ is the basic generator corresponding to the photon field, and the electric current j_{el}^μ is defined as

$$j_{\text{el}}^\mu \stackrel{\text{def}}{=} \bar{\psi} \gamma^\mu \psi \in \mathcal{P} \quad (7.2)$$

with the basic generators $\psi, \bar{\psi}$ corresponding to the electron and the positron field. It can be easily verified that this Lagrangian satisfies our requirements **(C1)** - **(C3)** and **(C5)**. The canonical dimension of the spinors is 3/2 and that of the photons is 1, summing up to a total canonical dimension of 4, so the model is renormalizable. We will show that also condition **(C4)** is accomplished.

In addition we examine an important relation that we were not able to prove in the general framework: The normalization condition **(N6)**. We prove that the other normalization conditions, in particular the Ward identities for the electric current, eqn. **(N5')**, already imply **(N6)** in QED. The proof will be given below.

To begin with we determine the $Q(n)$ -vertices of QED from its Lagrangian. For condition **(C4)** to hold we must find $Q(n)$ -vertices that satisfy the following equations:

$$s_0 T(\mathcal{L})(x) = i \partial_\nu^x T(\mathcal{L}_1^\nu)(x), \quad s_0 T(\mathcal{L}_1^\nu)(x) = i \partial_\rho^x T(\mathcal{L}_2^{\rho\nu})(x), \quad \dots \quad (7.3)$$

Observing the free BRS transformations introduced in section (3.5), we find that these $Q(n)$ -vertices exist indeed:

$$\mathcal{L}_1^\nu = u j_{\text{el}}^\nu, \quad \mathcal{L}_i = 0 \quad \forall i \geq 2. \quad (7.4)$$

To prove that **(N6)** is valid we must therefore calculate the following expression

$$\partial_\mu^y T \left(\mathcal{L}_1^{\nu_1}, \dots, \mathcal{L}_1^{\nu_k}, \mathcal{L}, \dots, \mathcal{L}, j_B^\mu \right) (x_1, \dots, x_n, y) \quad (7.5)$$

with $\mathcal{L} = \mathcal{L}_{QED}$, \mathcal{L}_1 like above and j_B^μ as defined in section (3.5). If we insert this time ordered product into the causal Wick expansion, eqn. (4.24), we find that it

can be written as

$$\begin{aligned} & T\left(\mathcal{L}_1^{\nu_1}, \dots, \mathcal{L}_1^{\nu_k}, \mathcal{L}, \dots, \mathcal{L}, (A_\rho)^{(1,\rho)}\right)(x_1, \dots, x_n, y) \cdot \partial_y^\mu u(y) \\ & - T\left(\mathcal{L}_1^{\nu_1}, \dots, \mathcal{L}_1^{\nu_k}, \mathcal{L}, \dots, \mathcal{L}, (A_\rho)^{(2,\rho\mu)}\right)(x_1, \dots, x_n, y) \cdot u(y). \end{aligned} \quad (7.6)$$

Since neither \mathcal{L} nor \mathcal{L}_1 contain a higher generator, conditions **(N4)** reveals that this expression is equal to

$$\begin{aligned} & \left(\partial_y^\rho T\left(\mathcal{L}_1^{\nu_1}, \dots, \mathcal{L}_1^{\nu_k}, \mathcal{L}, \dots, \mathcal{L}, (A_\rho)^{(0)}\right)(x_1, \dots, x_n, y)\right) \cdot \partial_y^\mu u(y) \\ & - \left(\partial_y^\rho \partial_y^\mu T\left(\mathcal{L}_1^{\nu_1}, \dots, \mathcal{L}_1^{\nu_k}, \mathcal{L}, \dots, \mathcal{L}, (A_\rho)^{(0)}\right)(x_1, \dots, x_n, y)\right) \cdot u(y). \end{aligned} \quad (7.7)$$

Inserting the derivative and taking into account the field equations of $u(y)$, we find

$$\begin{aligned} & \partial_\mu^y T\left(\mathcal{L}_1^{\nu_1}, \dots, \mathcal{L}_1^{\nu_k}, \mathcal{L}, \dots, \mathcal{L}, j_B^\mu\right)(x_1, \dots, x_n, y) \\ & = - \left(\partial_y^\mu \square^y T\left(\mathcal{L}_1^{\nu_1}, \dots, \mathcal{L}_1^{\nu_k}, \mathcal{L}, \dots, \mathcal{L}, (A_\mu)^{(0)}\right)(x_1, \dots, x_n, y)\right) \cdot u(y). \end{aligned} \quad (7.8)$$

With condition **(N4)** this expression can be rewritten as

$$\begin{aligned} & -i \left(\sum_{m=k+1}^n (\partial_\mu^y \delta(y - x_m)) \right. \\ & \quad \left. \times T\left(\mathcal{L}_1^{\nu_1}, \dots, \mathcal{L}_1^{\nu_k}, \mathcal{L}, \dots, j_{\text{el}}^\mu, \dots, \mathcal{L}\right)(x_1, \dots, x_n) \right) \cdot u(y). \end{aligned} \quad (7.9)$$

Here the vertex j_{el}^μ is at the m^{th} position. Pulling the derivative out of the bracket we finally arrive at

$$\begin{aligned} & i \sum_{m=k+1}^n \partial_\mu^m (\delta(y - x_m) T\left(\mathcal{L}_1^{\nu_1}, \dots, \mathcal{L}_1^{\nu_k}, \mathcal{L}, \dots, \mathcal{L}_1^\mu, \dots, \mathcal{L}\right)(x_1, \dots, x_n)) \\ & - i \left(\sum_{m=k+1}^n \delta(y - x_m) \partial_\mu^m T\left(\mathcal{L}_1^{\nu_1}, \dots, \mathcal{L}_1^{\nu_k}, \mathcal{L}, \dots, j_{\text{el}}^\mu, \dots, \mathcal{L}\right)(x_1, \dots, x_n) \right) \cdot u(y). \end{aligned} \quad (7.10)$$

The vertices \mathcal{L}_1^μ and j_{el}^μ are again in the m^{th} position. Comparing the last line with the Ward identities for the electric current, eqn. **(N5')**, we see that this term vanishes since $f(\mathcal{L}) = f(\mathcal{L}_1) = 0$. The remaining expression is exactly what condition **(N6)** predicts, provided that all the $R(n)$ -vertices vanish, $\mathcal{M}_1 = \mathcal{M}_2 = \dots = 0$. Condition **(N6)** was derived using the other normalization conditions, so it must be compatible with all these conditions.

Now we come to the definition of interacting fields. Since the Lagrangian contains no higher generators, the following relations hold due to eqn. (6.5)

$$\left((\varphi_i)^{(n,\nu_1 \dots \nu_n)}\right)_{\text{int}}^{g\mathcal{L}_{QED}}(x) = \partial_x^{\nu_1} \dots \partial_x^{\nu_n} \left((\varphi_i)^{(0)}\right)_{\text{int}}^{g\mathcal{L}_{QED}}(x). \quad (7.11)$$

We define $F^{\mu\nu}$ as

$$F^{\mu\nu} \stackrel{\text{def}}{=} (A^\nu)^{(1,\mu)} - (A^\mu)^{(1,\nu)} \in \mathcal{P}. \quad (7.12)$$

The easiest examples of interacting fields are the ghosts and the anti-ghosts. They do not appear in the Lagrangian \mathcal{L}_{QED} and therefore do not interact. The causal

Wick expansion, eqn. (4.24), implies together with the definition of the retarded products, eqn. (5.8),

$$(u)_{\text{int}}^{g\mathcal{L}_{QED}}(x) = u(x) \quad \text{and} \quad (\tilde{u})_{\text{int}}^{g\mathcal{L}_{QED}}(x) = \tilde{u}(x). \quad (7.13)$$

Due to relation (7.11) we can establish the usual relation for the interacting photon field and the field strength tensor in QED:

$$(F^{\mu\nu})_{\text{int}}^{g\mathcal{L}_{QED}}(x) = \partial_x^\mu (A_\nu)_{\text{int}}^{g\mathcal{L}_{QED}}(x) - \partial_x^\nu (A_\mu)_{\text{int}}^{g\mathcal{L}_{QED}}(x). \quad (7.14)$$

The field equations for QED are also the usual ones:

$$\begin{aligned} \square^x (A^\mu)_{\text{int}}^{g\mathcal{L}_{QED}}(x) &= -g(x) (j_{\text{el}}^\mu)_{\text{int}}^{g\mathcal{L}_{QED}}(x) \\ \text{and} \quad (i\partial\!\!\!/ - m) (\psi)_{\text{int}}^{g\mathcal{L}_{QED}}(x) &= -g(x) (\gamma^\mu A_\mu \psi)_{\text{int}}^{g\mathcal{L}_{QED}}(x). \end{aligned} \quad (7.15)$$

Furthermore we find that the interacting ghost current and BRS current have a particularly easy form because the ghosts and anti-ghosts do not interact:

$$\begin{aligned} (k^\mu)_{\text{int}}^{g\mathcal{L}_{QED}}(x) &= k^\mu(x) \\ \text{and} \quad (j_B^\mu)_{\text{int}}^{g\mathcal{L}_{QED}}(x) &= \left(\partial_x^\rho (A_\rho)_{\text{int}}^{g\mathcal{L}_{QED}}(x) \right) \partial_x^\mu u(x) \\ &\quad - \left(\partial_x^\rho \partial_x^\mu (A_\rho)_{\text{int}}^{g\mathcal{L}_{QED}}(x) \right) u(x). \end{aligned} \quad (7.16)$$

Dütsch and Fredenhagen [DF99] find the following commutator relations

$$\begin{aligned} \left[\partial_x^\mu (A_\mu)_{\text{int}}^{g\mathcal{L}_{QED}}(x), (A_\nu)_{\text{int}}^{g\mathcal{L}_{QED}}(y) \right]_- &= i\partial^\nu D(x-y) \\ \text{and} \quad \left[\partial_x^\mu (A_\mu)_{\text{int}}^{g\mathcal{L}_{QED}}(x), (\psi)_{\text{int}}^{g\mathcal{L}_{QED}}(y) \right]_- &= g(x)D(x-y) (\psi)_{\text{int}}^{g\mathcal{L}}(y) \end{aligned} \quad (7.17)$$

if $x, y \in \mathcal{O}$. We can use the equation for the interacting BRS current to find the explicit form of the interacting BRS transformations, for example

$$\begin{aligned} \tilde{s} \left((A_\mu)_{\text{int}}^{g\mathcal{L}_{QED}}(x) \right) &= i\partial_x^\mu u(x) & \tilde{s}(u(x)) &= 0 \\ \tilde{s} \left(\partial_x^\mu (A_\mu)_{\text{int}}^{g\mathcal{L}_{QED}}(x) \right) &= 0 & \tilde{s}(\tilde{u}(x)) &= -i\partial_x^\rho (A_\rho)_{\text{int}}^{g\mathcal{L}_{QED}}(x) \\ \tilde{s} \left((\psi)_{\text{int}}^{g\mathcal{L}_{QED}}(x) \right) &= -g(x) (\psi)_{\text{int}}^{g\mathcal{L}_{QED}}(x) u(x) & \tilde{s} \left((F^{\mu\nu})_{\text{int}}^{g\mathcal{L}_{QED}}(x) \right) &= 0 \\ \tilde{s} \left((\bar{\psi})_{\text{int}}^{g\mathcal{L}_{QED}}(x) \right) &= g(x) (\bar{\psi})_{\text{int}}^{g\mathcal{L}_{QED}}(x) u(x) & \tilde{s} \left((j_{\text{el}}^\mu)_{\text{int}}^{g\mathcal{L}_{QED}}(x) \right) &= 0, \end{aligned} \quad (7.18)$$

for $x \in \mathcal{O}$. The interacting electric current and the interacting field strength tensor are the only nontrivial observable quantities of those. The other two quantities with vanishing BRS transformation are not observable. The ghost $u(x)$ has non vanishing ghost number, and $\partial_x^\mu (A_\mu)_{\text{int}}^{g\mathcal{L}_{QED}}(x)$ is a coboundary and therefore equivalent to zero.

7.2. Yang-Mills-theory. The basic fields in Yang-Mills theory²⁰ are Lie algebra valued vector bosons $A_\mu = A_\mu^a \tau_a$, ghosts $u = u^a \tau_a$ and anti-ghosts $\tilde{u} = \tilde{u}^a \tau_a$. The τ_a form a basis of the Lie algebra. Their Lie-bracket gives $[\tau_a, \tau_b] = f_{ab}^c \tau_c$. The f_{ab}^c are the structure constants of the Lie-algebra. They satisfy the Jacobi-identity

$$f_{ab}^e f_{ec}^d + f_{bc}^e f_{ea}^d + f_{ca}^e f_{eb}^d = 0 \quad (7.19)$$

²⁰We consider here only pure, massless Yang-Mills theory, for simplicity

and are assumed to be totally antisymmetric.

The free field operators that belong to different components $A_\mu^a, u^a, \tilde{u}^a$ of the fields A_μ, u and \tilde{u} have trivial commutation relations among each other, e.g.

$$\{u^a(x), \tilde{u}^b(y)\}_+ = -i\delta^{ab}D(x-y). \quad (7.20)$$

Therefore the free model underlying Yang-Mills theory is simply a p -fold copy of free QED if p is the dimension of the Lie algebra. The underlying free model was considered in section (3.5)

The Lagrangian of Yang-Mills theory in causal perturbation theory is

$$\mathcal{L}_{YM} = \frac{1}{2}f_{ab}^c A_\mu^a A_\nu^b F_c^{\nu\mu} - f_{ab}^c A_\mu^b u^a \partial^\mu \tilde{u}_c. \quad (7.21)$$

Here $F_c^{\mu\nu} \stackrel{\text{def}}{=} (A_c^\nu)^{(1,\mu)} - (A_c^\mu)^{(1,\nu)}$. Note that there is no four-gluon-vertex present. It is created in second order perturbation theory due to $C_{A,1} = -\frac{1}{2}$, see [DHKS94a] - [DHS95b] for further details.

For the interacting fields we get

$$\begin{aligned} (A_\mu^a)_{\text{int}}^{g\mathcal{L}_{YM}}(x) &= \left((A_\mu^a)_{\text{int}}^{g\mathcal{L}_{YM}}(x) \right)^* && \in \tilde{\mathfrak{C}} \cdot \text{Dist}_1(\mathcal{D}), \\ (u^a)_{\text{int}}^{g\mathcal{L}_{YM}}(x) &= \left((u^a)_{\text{int}}^{g\mathcal{L}_{YM}}(x) \right)^* && \in \tilde{\mathfrak{C}} \cdot \text{Dist}_1(\mathcal{D}), \\ (\tilde{u}^a)_{\text{int}}^{g\mathcal{L}_{YM}}(x) &= - \left((\tilde{u}^a)_{\text{int}}^{g\mathcal{L}_{YM}}(x) \right)^* && \in \tilde{\mathfrak{C}} \cdot \text{Dist}_1(\mathcal{D}). \end{aligned} \quad (7.22)$$

From eqn. (6.5) we get for the higher generators

$$\begin{aligned} \left((A_\mu^a)^{(1,\nu)} \right)_{\text{int}}^{g\mathcal{L}_{YM}}(x) &= \partial_x^\nu (A_\mu^a)_{\text{int}}^{g\mathcal{L}_{YM}}(x) - \frac{1}{2}g(x) (f_{bc}^a A_\mu^b A_\nu^c)_{\text{int}}^{g\mathcal{L}_{YM}}(x), \\ \left((u^a)^{(1,\nu)} \right)_{\text{int}}^{g\mathcal{L}_{YM}}(x) &= \partial_x^\nu (u^a)_{\text{int}}^{g\mathcal{L}_{YM}}(x) + g(x) (f_{bc}^a A^{b,\nu} u^c)_{\text{int}}^{g\mathcal{L}_{YM}}(x). \\ \left((\tilde{u}^a)^{(1,\nu)} \right)_{\text{int}}^{g\mathcal{L}_{YM}}(x) &= \partial_x^\nu (\tilde{u}^a)_{\text{int}}^{g\mathcal{L}_{YM}}(x). \end{aligned} \quad (7.23)$$

The first equation implies in particular

$$(F_{\mu\nu}^a)_{\text{int}}^{g\mathcal{L}_{YM}}(x) = \partial_\mu^\nu (A_\nu^a)_{\text{int}}^{g\mathcal{L}_{YM}}(x) - \partial_\nu^\mu (A_\mu^a)_{\text{int}}^{g\mathcal{L}_{YM}}(x) + g(x) (f_{bc}^a A_\mu^b A_\nu^c)_{\text{int}}^{g\mathcal{L}_{YM}}(x) \quad (7.24)$$

and

$$\left((A_\mu^a)^{(1,\mu)} \right)_{\text{int}}^{g\mathcal{L}_{YM}}(x) = \partial_x^\mu (A_\mu^a)_{\text{int}}^{g\mathcal{L}_{YM}}(x). \quad (7.25)$$

The first relation reproduces the usual relation between the interacting vector boson field and the field strength tensor in Yang-Mills theories. From the Lagrangian we can also derive the field equations using eqn. (6.7):

$$\begin{aligned} \square^x (A_\mu^a)_{\text{int}}^{g\mathcal{L}_{YM}}(x) &= \partial_x^\nu \left[g(x) (f_{bc}^a A_\mu^b A_\nu^c)_{\text{int}}^{g\mathcal{L}_{YM}}(x) \right] \\ &\quad - g(x) (f_{bc}^a A^{\nu,b} F_{\nu\mu}^c)_{\text{int}}^{g\mathcal{L}_{YM}}(x) + g(x) (f_{bc}^a u^b (\tilde{u}^c)_{(1,\mu)})_{\text{int}}^{g\mathcal{L}_{YM}}(x), \\ \square^x (u^a)_{\text{int}}^{g\mathcal{L}_{YM}}(x) &= -\partial_x^\mu \left[g(x) (f_{bc}^a A^{\mu,b} u^c)_{\text{int}}^{g\mathcal{L}_{YM}}(x) \right], \\ \square^x (\tilde{u}^a)_{\text{int}}^{g\mathcal{L}_{YM}}(x) &= -g(x) \left(f_{bc}^a A_\mu^b (\tilde{u}^c)^{(1,\mu)} \right)_{\text{int}}^{g\mathcal{L}_{YM}}(x). \end{aligned} \quad (7.26)$$

The Lagrangian (7.21) obviously satisfies the conditions **(C1)**, **(C2)**, **(C3)** and **(C5)**. It is also possible to find $Q(n)$ -vertices and $R(n)$ -vertices such that condition **(C4)** is valid. These vertices are

$$\begin{aligned} \mathcal{L}_1^\mu &= f_{ab}^c u^a A_\nu^b F_c^{\nu\mu} - \frac{1}{2} f_{ab}^c u^a u^b (\tilde{u}_c)^{(1,\mu)}, & \mathcal{M}_1^\mu &= \frac{1}{2} f_{ab}^c u^a u^b (\tilde{u}_c)^{(1,\mu)}, \\ \mathcal{L}_2^{\mu\rho} &= \frac{1}{2} f_{ab}^c u^a u^b F_c^{\mu\rho}, & \mathcal{M}_2^{\mu\rho} &= \frac{1}{2} f_{ab}^c u^a u^b F_c^{\mu\rho}, \\ \mathcal{L}_{i\cdots} &= \mathcal{M}_{i\cdots} = 0 \quad \forall i > 2. \end{aligned} \quad (7.27)$$

The vertices have been chosen such that condition **(N6)** is compatible with all other normalization conditions in first order.

We remind the reader that the existence of solutions for condition **(N6)** has not been proven for an arbitrary number of arguments in Yang-Mills theory. Scharf and collaborators, [DHKS94a] - [DHS95b], have proven operator gauge invariance, i.e. eqn. (4.35) for $i_1 = \cdots = i_n = 0$, see section (4.4) for further details.

We already mentioned that the free model underlying Yang-Mills theory is a p -fold copy of free QED if p is the dimension of the Lie-group. The question arises whether there are other interactions besides Yang-Mills theory with the same free model. Stora [Sto97] found out that the number of possible Lagrangians for such a model is severely restricted by the conditions **(C1)** - **(C5)**. Lagrangians $T(\mathcal{L})(x)$ may differ from the Yang-Mills Lagrangian only by a coboundary $s_0 T(K)(x)$ or a derivative $\partial_\mu T(K^\mu)(x)$ where K is a scalar polynomial with ghost number -1 and K^μ is a vector polynomial with ghost number zero. By Yang-Mills Lagrangian we mean here an expression like (7.21) with arbitrary constants f_{bc}^a that are totally antisymmetric in their indices and satisfy the Jacobi identity (7.19). In particular the Lie-group structure needs not to be put in. The Jacobi identity for the constants f_{bc}^a is a consequence of operator gauge invariance in second order, and operator gauge invariance in first order implies that they are totally antisymmetric. Since Stora's paper is not published, we refer the reader to the articles of Aste and Scharf [AS98] and Grigore [Gri98].

It is usually argued that the addition of such coboundary or derivative terms does not change the model because coboundaries are equivalent to zero in cohomology and derivatives should not give a contribution in the adiabatic limit. Dütsch [Düt96] has proven that this is correct also in higher orders for theories where the adiabatic limit can be performed, e.g. in massive theories. But for models where this limit does not exist the question is still open. Concerning the coboundary terms we remark that it is not clear whether a coboundary in the free theory, $T(A) = s_0 T(B)$ for some $B \in \mathcal{P}$, gives a coboundary in the interacting theory, such that $(A)_{\text{int}}^{g\mathcal{L}}(x) = \tilde{s}(C)_{\text{int}}^{g\mathcal{L}}(x)$ for some $C \in \mathcal{P}$. Direct calculations in first order indicate that this is indeed true for suitable normalizations, but as long as this question is not clarified coboundary terms in the Lagrangian must not be neglected. The same is true for derivated terms in these theories.

In the rest of the section we want to compare our results with those of Nakanishi and Ojima [NO90]. Their results have been derived in the context of quantum field theory, but they are also classical in the following sense: They use field equations derived as Euler-Lagrange equations from a classical action, and they deliberately neglect the distributional character of field operators and form products of field operators at the same spacetime point. Therefore it is possible to compare their results with the classical limit of our results. At first we note that Nakanishi and

Ojima use a different convention for the anti-ghosts. Their ghosts C and anti-ghosts \bar{C} correspond to ours in the following way:

$$C^a \longleftrightarrow u^a, \quad \bar{C}^a \longleftrightarrow i\tilde{u}^a. \quad (7.28)$$

For the comparison we will always translate their results into our language. To make the notation shorter we introduce the covariant derivative of a field with Lie-algebra index, $X(x) = X^a(x)\tau_a$, as

$$(D_\mu X)^a(x) \stackrel{\text{def}}{=} \partial_\mu^x X^a(x) + f_{bc}^a (A_\mu^b)_{\text{cl}}(x) X^c(x). \quad (7.29)$$

We have the classical fields

$$C(A_\mu^a)(x) = (A_\mu^a)_{\text{cl}}(x), \quad C(u^a)(x) = (u^a)_{\text{cl}}(x), \quad C(\tilde{u}^a)(x) = (\tilde{u}^a)_{\text{cl}}(x) \quad (7.30)$$

and for the higher generators the representation C gives

$$\begin{aligned} C\left((A_\mu^a)^{(1,\nu)}\right)(x) &= \partial_x^\nu (A_\mu^a)_{\text{cl}}(x) - \frac{1}{2}g(x)f_{bc}^a (A_\mu^b)_{\text{cl}}(x) (A_\nu^c)_{\text{cl}}(x), \\ C\left((u^a)^{(1,\nu)}\right)(x) &= \partial_x^\nu (u^a)_{\text{cl}}(x) + g(x)f_{bc}^a (A^{b,\nu})_{\text{cl}}(x) (u^c)_{\text{cl}}(x), \\ C\left((\tilde{u}^a)^{(1,\nu)}\right)(x) &= \partial_x^\nu (\tilde{u}^a)_{\text{cl}}(x). \end{aligned} \quad (7.31)$$

The field equations (7.26) become in the classical limit

$$\begin{aligned} (D^\mu F_{\mu\nu}^{\text{cl}})^a(x) &= -\partial_\nu^x \partial_x^\mu (A_\mu^a)_{\text{cl}}(x) \\ &\quad + g f_{bc}^a (\partial_\nu^x (\tilde{u}^b)_{\text{cl}}(x)) \cdot (u^c)_{\text{cl}}(x), \\ \partial_x^\mu (D_\mu (u)_{\text{cl}})^a(x) &= 0, \\ (D_\mu \partial^\mu (u)_{\text{cl}})^a(x) &= 0. \end{aligned} \quad (7.32)$$

Here $F_{\mu\nu}^{a,\text{cl}}$ is the classical field strength tensor,

$$F_{\mu\nu}^{a,\text{cl}} = \partial_\mu^x (A_\nu^a)_{\text{cl}}(x) - \partial_\nu^x (A_\mu^a)_{\text{cl}}(x) + g f_{bc}^a (A_\mu^b)_{\text{cl}}(x) (A_\nu^c)_{\text{cl}}(x). \quad (7.33)$$

The field equations are exactly the same as those of Nakanishi and Ojima. For the ghost current we get

$$C(k^\mu)(x) = i \sum_a \left((u^a)_{\text{cl}}(x) \partial_x^\mu (\tilde{u}^a)_{\text{cl}}(x) - (D^\mu (u)_{\text{cl}})^a(x) (\tilde{u}^a)_{\text{cl}}(x) \right). \quad (7.34)$$

This is $-i$ times the result of Nakanishi and Ojima. The factor $-i$ comes from a different definition of the ghost current. They require that the ghost current and -charge be pseudo-hermitian, so that the eigenvalues of the ghost charge are in $i\mathbb{Z}$. For the classical BRS current $j_B^\mu(x)$ we have according to definition (6.26)

$$j_B^\mu(x) = (j_B^\mu)_{\text{cl}}(x) - g (\mathcal{M}_1^\mu)_{\text{cl}}(x). \quad (7.35)$$

This reads in terms of the basic fields

$$\begin{aligned} j_B^\mu(x) &= \sum_a \left((D^\mu (u)_{\text{cl}})^a(x) \partial_x^\nu (A_\nu^a)_{\text{cl}}(x) - (u^a)_{\text{cl}}(x) \partial_x^\mu \partial_x^\nu (A_\nu^a)_{\text{cl}}(x) \right) \\ &\quad - \frac{1}{2} f_{ab}^c (u^a)_{\text{cl}}(x) (u^b)_{\text{cl}}(x) \partial_x^\mu (\tilde{u}^c)_{\text{cl}}(x) \end{aligned} \quad (7.36)$$

This is again — up to a minus sign which is pure convention — the same result as Nakanishi and Ojima. Therefore we realize a complete agreement between the results of Nakanishi and Ojima and ours, apart from different conventions. This supports both our results at the quantum level and also the correspondence law.

The same relations at the quantum level would have given different results if we had adopted the correspondence law (6.53), for example.

8. CONCLUSIONS AND OUTLOOK

We presented a universal construction of local quantum gauge theories. It gives an algebra of local observables that has a Hilbert space representation. For this construction to work two preconditions must hold: The underlying free theory must be positive in the sense discussed in chapter (3), and the time ordered products of free field operators must satisfy the conditions **(N1)** - **(N6)**. The second precondition can only be violated with respect to condition **(N6)**, all other conditions can always be accomplished. If all the normalization conditions hold, a locally conserved BRS current and with it a nilpotent BRS transformation on the algebra of local fields can be defined. The algebra of local observables is then defined as the cohomology of the algebra of local fields w.r.t. the BRS transformation. If the underlying free model is positive, the Hilbert space representation can be constructed. Therefore spacetime must be compactified spatially in order to allow a nilpotent BRS charge to be defined. This compactification does not change the algebra. It is an open question whether two representations that are constructed with a different compactification length are equivalent or not.

The most crucial point for each model that is investigated in this framework is whether normalization condition **(N6)** can be accomplished together with the other normalization conditions. We have proven that this holds for quantum electrodynamics, but for Yang-Mills theory the question is still open. We think that methods of algebraic renormalization can help to find a solution. To reduce the problem to an algebraic one it could be helpful to define a BRS transformation s on the algebra \mathcal{P} , such that $T(sA) = s_0 T(A) \quad \forall A \in \mathcal{P}$. This requires the introduction of an additional auxiliary field, the scalar Nakanishi-Lautrup field $B \in \mathcal{P}$ with the properties $s\tilde{u} = iB$, $sB = 0$ and $T(B)(x) = -\partial^\mu A_\mu(x)$. With this definition the BRS transformation s on \mathcal{P} can be chosen to be nilpotent. These notions could make it possible to translate the language of algebraic renormalization into ours. Normalization condition **(N6)** takes on the form of the descent equations in algebraic renormalization. Since they are proven in Yang-Mills theory, this could also lead to a proof of **(N6)** for Yang-Mills theory.

The renormalization scheme underlying our construction is the one of Epstein and Glaser. It is formulated, unlike the other renormalization schemes, in configuration space. Therefore it is suitable for quantum field theories on curved spacetimes. Brunetti and Fredenhagen [BF99] have shown that the time ordered products can also be defined in globally hyperbolic spacetimes. To generalize our normalization conditions to these spacetimes, the propagators and differential operators introduced in chapter (3) must be substituted by suitably generalized ones. With the normalization conditions all relations derived from them carry over to curved spacetimes, in particular the field equations, the conservation of ghost and BRS current and the nilpotency of the BRS charge and the BRS transformation. So it is possible to define an algebra of local observables even in globally hyperbolic spacetimes, provided these spacetimes allow propagators and their corresponding differential operators to be defined.

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APPENDIX A. PROOF OF (N3) AND (N4)

A.1. **Proof of (N3).** The essential point in the proof that solutions for condition (N3) exist is to show that eqn. (N3) is equivalent to the causal Wick expansion (4.24). This suffices for a proof because it was already shown in [BF99] that (4.24) has solutions, see below.

The proof that both conditions are equivalent for a certain $T(W_1, \dots, W_n)$ proceeds inductively. The induction hypothesis is that eqn. (N3) and eqn. (4.24) hold and are equivalent for the following time ordered products: all time ordered products that contain fewer arguments than n and all that contain a combination of sub monomials of the W_i , if at least one of these sub monomials is a proper one.

At first we prove that eqn. (4.24) implies eqn. (N3). With eqn. (4.24) the time ordered product on the left hand side of (N3) can be written as

$$T(W_1, \dots, W_n)(x_1, \dots, x_n) = \sum_{\gamma_1, \dots, \gamma_n} \omega_0 \left(T(W_1^{(\gamma_1)}, \dots, W_n^{(\gamma_n)})(x_1, \dots, x_n) \right) \times \frac{:\varphi^{\gamma_1}(x_1) \cdots \varphi^{\gamma_n}(x_n):}{\gamma_1! \cdots \gamma_n!}, \quad (\text{A.1})$$

for the notation see the formulas following (4.24). To calculate the (anti-) commutator with the $\varphi_i(z)$ in eqn. (N3), we note that the (anti-) commutator of the Wick product with the $\varphi_i(z)$ gives

$$\left[\frac{:\varphi^{\gamma_1}(x_1) \cdots \varphi^{\gamma_n}(x_n):}{\gamma_1! \cdots \gamma_n!}, \varphi_i(z) \right]_{\mp} = i \sum_{k=1}^n \sum_j \Delta_{ij}(z - x_k) \frac{:\varphi^{\gamma_1}(x_1) \cdots \varphi^{\gamma_k - e_j}(x_k) \cdots \varphi^{\gamma_n}(x_n):}{\gamma_1! \cdots (\gamma_k - e_j)! \cdots \gamma_n!} \quad (\text{A.2})$$

if the $\gamma_k \neq 0$, otherwise the respective term vanishes. Here e_j is the unit vector with an entry 1 at the j^{th} position and the other entries zero. Therefore we get for the complete commutator

$$i \sum_{k=1}^n \sum_j \Delta_{ij}(z - x_k) \sum_{\substack{\gamma_1, \dots, \gamma_n \\ \gamma_k \neq 0}} \omega_0 \left(T(W_1^{(\gamma_1)}, \dots, W_n^{(\gamma_n)})(x_1, \dots, x_n) \right) \times \left[\frac{:\varphi^{\gamma_1}(x_1) \cdots \varphi^{\gamma_k - e_j}(x_k) \cdots \varphi^{\gamma_n}(x_n):}{\gamma_1! \cdots (\gamma_k - e_j)! \cdots \gamma_n!} \right]. \quad (\text{A.3})$$

This becomes after a shifting of indices

$$\begin{aligned} & i \sum_{k=1}^n \sum_j \Delta_{ij}(z - x_k) \times \\ & \times \sum_{\gamma_1, \dots, \gamma_n} \omega_0 \left(T(W_1^{(\gamma_1)}, \dots, W_k^{(\gamma_1 + e_j)}, \dots, W_n^{(\gamma_n)})(x_1, \dots, x_n) \right) \times \\ & \times \left[\frac{:\varphi^{\gamma_1}(x_1) \cdots \varphi^{\gamma_n}(x_n):}{\gamma_1! \cdots \gamma_n!} \right] \\ & = i \sum_{k=1}^n \sum_j \Delta_{ij}(z - x_k) T(W_1, \dots, W_k^{(e_j)}, \dots, W_n)(x_1, \dots, x_n). \end{aligned} \quad (\text{A.4})$$

The last identity is valid because eqn. (A.1) holds for the sub monomials according to our induction hypothesis. This proves that **(N3)** is a consequence of (A.1). To complete the proof of equivalence we recall that we already saw that eqn. **(N3)** determines the time ordered product up to a \mathbf{C} -number distribution. To be precise, eqn. **(N3)** determines completely

$$T(W_1, \dots, W_n) - \omega_0(T(W_1, \dots, W_n)) \mathbb{1} \quad (\text{A.5})$$

and leaves

$$\omega_0(T(W_1, \dots, W_n)) \quad (\text{A.6})$$

open. This is exactly the same with (A.1). The Wick products are determined anyway and the numerical distributions are determined by the T -products for the sub monomials if at least one $\gamma_i \neq 0$. Since both equations determine the same part of the distribution and leave the same part open, and moreover one of them is a consequence of the other, they must be equivalent.

The question arises whether the expression on the right hand side of eqn. (A.1) is well defined, because there appear products of distribution. The answer is the same as in section (4.2): Epstein and Glaser's "Theorem 0" guarantees that the product is well defined.

A.2. Proof of (N4). Like for **(N3)** we do not prove the existence of solutions for **(N4)** itself but for its integrated version

$$\begin{aligned} T(W_1, \dots, W_n, \varphi_i)(x_1, \dots, x_n, y) &= \\ &= i \sum_{k=1}^n \sum_j \Delta_{ij}^F(y - x_k) T\left(W_1, \dots, \frac{\partial W_k}{\partial \varphi_j}, \dots, W_n\right)(x_1, \dots, x_n) \\ &\quad + \sum_{\gamma_1 \dots \gamma_n} \omega_0\left(T\left(W_1^{(\gamma_1)}, \dots, W_n^{(\gamma_n)}\right)(x_1, \dots, x_n)\right) \frac{:\varphi^{\gamma_1}(x_1) \dots \varphi^{\gamma_n}(x_n) \varphi_i(y):}{\gamma_1! \dots \gamma_n!}. \end{aligned} \quad (\text{A.7})$$

At the end of the section we will prove that the two conditions are equivalent.

The right hand side of eqn. (A.7) is obviously well defined, because the first sum is a tensor product of distributions which is always well defined — the argument y does not appear in the time ordered product — while the second sum is simply part of (A.1) which was already proven to be well defined.

The question is whether this expression has the correct causal factorization outside the diagonal. To show this we proceed again inductively, the induction hypothesis is that eqn. (A.7) is valid for all time ordered products of sub monomials of the W_i .

At first we compare the expression with (A.1), which reveals in the present case

$$\begin{aligned} T(W_1, \dots, W_n, \varphi_i)(x_1, \dots, x_n, y) &= \\ &= \sum_{\gamma_1 \dots \gamma_n} \omega_0\left(T\left(W_1^{(\gamma_1)}, \dots, W_n^{(\gamma_n)}, \varphi_i\right)(x_1, \dots, x_n, y)\right) \frac{:\varphi^{\gamma_1}(x_1) \dots \varphi^{\gamma_n}(x_n):}{\gamma_1! \dots \gamma_n!} \\ &\quad + \sum_{\gamma_1 \dots \gamma_n} \omega_0\left(T\left(W_1^{(\gamma_1)}, \dots, W_n^{(\gamma_n)}\right)(x_1, \dots, x_n)\right) \frac{:\varphi^{\gamma_1}(x_1) \dots \varphi^{\gamma_n}(x_n) \varphi_i(y):}{\gamma_1! \dots \gamma_n!}. \end{aligned} \quad (\text{A.8})$$

So the second sum in eqn. (A.7) is already present and we must only show that

$$i \sum_{k=1}^n \sum_j \Delta_{ij}^F(y - x_k) T \left(W_1, \dots, \frac{\partial W_k}{\partial \varphi_j}, \dots, W_n \right) (x_1, \dots, x_n) \quad (\text{A.9})$$

is a possible extension of

$$\sum_{\gamma_1 \cdots \gamma_n} \omega_0 \left(T^0 \left(W_1^{(\gamma_1)}, \dots, W_n^{(\gamma_n)}, \varphi_i \right) (x_1, \dots, x_n, y) \right) \frac{:\varphi^{\gamma_1}(x_1) \cdots \varphi^{\gamma_n}(x_n):}{\gamma_1! \cdots \gamma_n!} \quad (\text{A.10})$$

to the diagonal. Inserting eqn. (N3) into expression (A.9) gives

$$i \sum_{k=1}^n \sum_j \Delta_{ij}^F(y - x_k) \times \left[\sum_{\gamma_1 \cdots \gamma_n} \omega_0 \left(T \left(W_1^{(\gamma_1)}, \dots, W_k^{(\gamma_k + e_j)}, \dots, W_n^{(\gamma_n)} \right) (x_1, \dots, x_n) \right) \times \frac{:\varphi^{\gamma_1}(x_1) \cdots \varphi^{\gamma_n}(x_n):}{\gamma_1! \cdots \gamma_n!} \right]. \quad (\text{A.11})$$

The latter is equal to expression (A.10) if

$$i \sum_{k=1}^n \sum_j \Delta_{ij}^F(y - x_k) \times \left[\sum_{\gamma_1 \cdots \gamma_n} \omega_0 \left(T \left(W_1^{(\gamma_1)}, \dots, W_k^{(\gamma_k + e_j)}, \dots, W_n^{(\gamma_n)} \right) (x_1, \dots, x_n) \right) \right] = \omega_0 \left(T^0 \left(W_1^{(\gamma_1)}, \dots, W_n^{(\gamma_n)}, \varphi_i \right) (x_1, \dots, x_n, y) \right) \quad (\text{A.12})$$

for all $\gamma_1, \dots, \gamma_n$ and outside the diagonal. This equation is obviously true if at least one $\gamma_i \neq 0$ since eqn. (A.7) is valid for the sub monomials of the W_i according to our induction hypothesis. So eqn. (A.7) can be accomplished if

$$i \sum_{k=1}^n \sum_j \Delta_{ij}^F(y - x_k) \omega_0 \left(T \left(W_1, \dots, W_k^{(e_j)}, \dots, W_n \right) (x_1, \dots, x_n) \right) \quad (\text{A.13})$$

is a possible extension of

$$\omega_0 \left(T^0 \left(W_1, \dots, W_n, \varphi_i \right) (x_1, \dots, x_n, y) \right). \quad (\text{A.14})$$

To see this we smear out both expressions with a test function η that vanishes with all its derivatives on the diagonal Δ_{n+1} . Let the test function η be fix and recall the definition of the partition of unity in eqn. (4.18). Then we can define for each subset $Z \subset \{x_1, \dots, x_n, y\}$ another test function $\eta_Z \in \mathcal{D}(M^{n+1})$

$$\eta_Z \stackrel{\text{def}}{=} \begin{cases} f_Z \cdot \eta & \text{outside } \Delta_{n+1} \\ 0 & \text{otherwise} \end{cases} \quad (\text{A.15})$$

such that

$$\text{supp } \eta_Z \in \mathfrak{C}_Z \quad \text{and} \quad \sum_Z \eta_Z = \eta. \quad (\text{A.16})$$

Then the following equation is valid owing to causal factorization:

$$\begin{aligned}
& \int d^4 y d^4 x_1 \cdots d^4 x_n \eta(x_1, \dots, x_n, y) T(W_1, \dots, W_n, \phi_i)(x_1, \dots, x_n, y) = \\
& = \sum_{Z \subset X} \int d^4 y d^4 x_1 \cdots d^4 x_n \eta_Z(x_1, \dots, x_n, y) T(W_Z)(x_Z) T(W_{Z^c}, \varphi_i)(x_{Z^c}, y) \\
& \quad + \sum_{Z \subset X} \int d^4 y d^4 x_1 \cdots d^4 x_n \eta_Z(x_1, \dots, x_n, y) T(W_Z, \varphi_i)(x_Z, y) T(W_{Z^c})(x_{Z^c})
\end{aligned} \tag{A.17}$$

because $Z \supseteq Z^c$ on $\text{supp } \eta_Z$. Here $X = \{x_1, \dots, x_n\}$.

Let us investigate $T(W_Z)(x_Z) T(W_{Z^c}, \varphi_i)(x_{Z^c}, y)$ and assume for simplicity that $Z = \{x_{k+1}, \dots, x_n\}$ and $Z^c = \{x_1, \dots, x_k\}$. Due to the validity of eqn. (A.7) in lower orders we have

$$\begin{aligned}
& T(W_Z)(x_Z) T(W_{Z^c}, \varphi_i)(x_{Z^c}, y) = \\
& = i \sum_{m=1}^k \sum_j \Delta_{ij}^F(y - x_k) T(W_Z)(x_Z) T(W_1, \dots, W_m^{(e_j)}, \dots, W_k)(x_{Z^c}) \\
& \quad + \sum_{\gamma_1 \cdots \gamma_k} \omega_0 \left(T(W_1^{(\gamma_1)}, \dots, W_k^{(\gamma_k)})(x_{Z^c}) \right) T(W_Z)(x_Z) \times \\
& \quad \quad \quad \times \frac{:\varphi^{\gamma_1}(x_1) \cdots \varphi^{\gamma_k}(x_k) \varphi_i(y):}{\gamma_1! \cdots \gamma_k!}.
\end{aligned} \tag{A.18}$$

Since $Z \supseteq Z^c$, the product in the first sum recombines to

$$\begin{aligned}
& T(W_Z)(x_Z) T(W_1, \dots, W_m^{(e_j)}, \dots, W_k)(x_{Z^c}) \\
& = T(W_1, \dots, W_m^{(e_j)}, \dots, W_n)(x_1, \dots, x_n).
\end{aligned} \tag{A.19}$$

For the product in the second sum we get

$$\begin{aligned}
& T(W_Z)(x_Z) \frac{:\varphi^{\gamma_1}(x_1) \cdots \varphi^{\gamma_k}(x_k) \varphi_i(y):}{\gamma_1! \cdots \gamma_k!} = \\
& \quad : \dots \varphi_i(y) : + \sum_{l=k+1}^n \sum_j T(W_{k+1}, \dots, W_l^{(e_j)}, \dots, W_n)(x_Z) \Delta_{ij}^+(y - x_l) \\
& \quad \quad \quad \times \frac{:\varphi^{\gamma_1}(x_1) \cdots \varphi^{\gamma_k}(x_k):}{\gamma_1! \cdots \gamma_k!}.
\end{aligned} \tag{A.20}$$

Inserting this into eqn. (A.18) and taking (A.1) into account, the second sum becomes

$$\begin{aligned}
& i \sum_{l=1}^k \sum_j \Delta_{ij}^+(y - x_k) T(W_{k+1}, \dots, W_l^{(e_j)}, \dots, W_n)(x_Z) T(W_{Z^c})(x_{Z^c}) \\
& \quad + : \dots \varphi_i(y) :.
\end{aligned} \tag{A.21}$$

From the definition of the Feynman propagator we find

$$\Delta_{ij}^+(y - x_l) = \Delta_{ij}^F(y - x_l) - \Delta_{ij}^A(y - x_l) = \Delta_{ij}^F(y - x_l) \tag{A.22}$$

since $(y - x_l) \neq \bar{V}_-$. Recombining the terms we finally arrive at

$$\begin{aligned}
& \omega_0 \left(T \left(W_Z \right) (x_Z) T \left(W_{Z^c}, \varphi_i \right) (x_{Z^c}, y) \right) = \\
& = i \sum_{m=1}^k \sum_j \Delta_{ij}^F (y - x_k) \omega_0 \left(T \left(W_1, \dots, W_m^{(e_j)}, \dots, W_n \right) (x_1, \dots, x_n) \right) \\
& + i \sum_{l=k+1}^n \sum_j \Delta_{ij}^F (y - x_l) \omega_0 \left(T \left(W_1, \dots, W_l^{(e_j)}, \dots, W_n \right) (x_1, \dots, x_n) \right) \\
& = i \sum_{m=1}^n \sum_j \Delta_{ij}^F (y - x_k) \omega_0 \left(T \left(W_1, \dots, W_m^{(e_j)}, \dots, W_n \right) (x_1, \dots, x_n) \right).
\end{aligned} \tag{A.23}$$

With the same argument we can see that

$$\begin{aligned}
& \omega_0 \left(T \left(W_Z, \varphi_i \right) (x_Z, y) T \left(W_{Z^c} \right) (x_{Z^c}) \right) = \\
& = i \sum_{m=1}^n \sum_j \Delta_{ij}^F (y - x_k) \omega_0 \left(T \left(W_1, \dots, W_m^{(e_j)}, \dots, W_n \right) (x_1, \dots, x_n) \right).
\end{aligned} \tag{A.24}$$

Taking the vacuum expectation value of eqn. (A.17) and inserting the expressions above, we finally get

$$\begin{aligned}
& \int d^4 y d^4 x_1 \cdots d^4 x_n \eta(x_1, \dots, x_n, y) \omega_0 \left(T \left(W_1, \dots, W_n, \phi_i \right) (x_1, \dots, x_n, y) \right) \\
& = i \int d^4 y d^4 x_1 \cdots d^4 x_n \eta(x_1, \dots, x_n, y) \times \\
& \quad \times \sum_{m=1}^n \sum_j \Delta_{ij}^F (y - x_k) \omega_0 \left(T \left(W_1, \dots, W_m^{(e_j)}, \dots, W_n \right) (x_1, \dots, x_n) \right).
\end{aligned} \tag{A.25}$$

So we have proven that expression (A.13) is a possible extension of (A.14), and this implies that eqn. (A.7) has the correct causal factorization. From the construction it is clear that (A.7) is compatible with (4.24) and thus with **(N3)**. It is obvious that it respects the Poincaré transformation properties and is therefore compatible with **(N1)**. The same calculation as for the compatibility of eqns. **(N1)** and **(N2)** reveals that it is also compatible with **(N2)**.

Finally we have to prove that **(N4)** and (A.7) are equivalent. Eqn. (A.7) implies **(N4)** immediately: Application of the operator D^y , eqn. (3.86), from the left on eqn. (A.7) gives the desired result.

On the other hand a solution of **(N4)** is unique. This can best be seen for the corresponding equation for the retarded products,

$$\begin{aligned}
& \sum_j D_{ij}^y R \left(W_1, \dots, W_n; \varphi_j \right) (x_1, \dots, x_n; y) = \\
& = i \sum_{k=1}^n R \left(W_1, \dots, \check{k}, \dots, W_n; \frac{\partial W_k}{\partial \varphi_i} \right) (x_1, \dots, \check{k}, \dots, x_n; x_k) \delta(x_k - y).
\end{aligned} \tag{N4}$$

The difference of two solutions of this differential equation is a solution of the homogeneous differential equation. Due to the support properties of the retarded products there exists a Cauchy surface in the y -space such that all Cauchy data are zero. Therefore zero is a solution of that equation, and D^y is an operator with a unique solution for the Cauchy problem, see page 29. So the retarded products are uniquely determined and with them the time ordered products. This completes the proof that **(N4)** and (A.7) are equivalent.

APPENDIX B. PROOFS CONCERNING THE WARD IDENTITIES

This appendix contains in its first section the proof that the ghost number Ward identities have common solutions with the other normalization conditions and that the ghost number Ward identities imply eqn. (4.32). In the second section we prove that the validity of the generalized operator gauge invariance already implies that there exists a solution of condition **(N6)**.

B.1. Proof of the ghost number Ward identities. We begin with the proof that the equation

$$\begin{aligned} s_c T(W_1 \cdots W_n)(x_1, \dots, x_n) &= \\ &= \left(\sum_{k=1}^n g(W_k) \right) T(W_1 \cdots W_n)(x_1, \dots, x_n). \end{aligned} \quad (\text{B.1})$$

is a direct consequence of condition **(N5)**,

$$\begin{aligned} \partial_\mu^y T(W_1, \dots, W_n, k^\mu)(x_1, \dots, x_n, y) &= \\ &= \sum_{k=1}^n g(W_k) \delta(y - x_k) T(W_1, \dots, W_n)(x_1, \dots, x_n). \end{aligned} \quad (\text{B.2})$$

Suppose, \mathcal{O} is an open, bounded and causally complete region in spacetime such that all points x_1, \dots, x_n in eqn. **(N5)** lie in \mathcal{O} — obviously for every set of points such a region can be found. Then we choose a test function $f \in \mathcal{D}(M)$ such that $f(x) = 1 \quad \forall x \in \mathcal{O}'$ with \mathcal{O}' another open, bounded and causally complete region such that $\overline{\mathcal{O}} \subset \mathcal{O}'$. Then we can find a Lorentz frame where a C^∞ -function $H(y)$ exists with the following properties:

$$\begin{aligned} H &\in C^\infty(M), \quad \exists H^t \in C^\infty(\mathbb{R}) : \quad H(y) = H^t(y^0), \\ H^t(y^0) &= 1 \quad \forall y^0 < -\epsilon, \quad H^t(y^0) = 0 \quad \forall y^0 > \epsilon, \quad \epsilon \in \mathbb{R}, \quad 0 < \epsilon \ll 1, \\ \text{supp}(H \cdot \partial_\mu f) \cap (\overline{V}_+ + \mathcal{O}) &= \emptyset, \quad \text{supp}((1 - H) \cdot \partial_\mu f) \cap (\overline{V}_- + \mathcal{O}) = \emptyset. \end{aligned} \quad (\text{B.3})$$

The following calculations will be done in that Lorentz frame. Smearing out the left hand side of eqn. **(N5)** with f gives

$$\begin{aligned} \int d^4y f(y) \partial_\mu^y T(W_1, \dots, W_n, k^\mu)(x_1, \dots, x_n, y) &= \\ &= - \int d^4y (\partial_\mu f)(y) \cdot H(y) \cdot T(W_1, \dots, W_n, k^\mu)(x_1, \dots, x_n, y) \\ &\quad - \int d^4y (\partial_\mu f)(y) \cdot (1 - H(y)) \cdot T(W_1, \dots, W_n, k^\mu)(x_1, \dots, x_n, y). \end{aligned} \quad (\text{B.4})$$

According to our assumptions about the supports of the test functions $(\partial_\mu f) \cdot H(y)$ and $(\partial_\mu f) \cdot (1 - H(y))$ we have in the first integral on the right hand side $y \gtrsim x_i \quad \forall i$ and in the second integral on the right hand side $x_i \gtrsim y \quad \forall i$. Owing to causal factorization the time ordered product $T(W_1, \dots, W_n, k^\mu)(x_1, \dots, x_n, y)$ decomposes in the first integral according to $T(k^\mu)(y)T(W_1, \dots, W_n)(x_1, \dots, x_n)$ and in the second one according to $T(W_1, \dots, W_n)(x_1, \dots, x_n)T(k^\mu)(y)$. Therefore the integral

can be written as

$$\begin{aligned}
& \int d^4y f(y) \partial_\mu^y T(W_1, \dots, W_n, k^\mu)(x_1, \dots, x_n, y) = \\
& = \int d^4y (\partial_\mu f)(y) \cdot H(y) \cdot \left[T(k^\mu)(y), T(W_1, \dots, W_n)(x_1, \dots, x_n) \right]_{\mp} \quad (\text{B.5}) \\
& - \int d^4y (\partial_\mu f)(y) \cdot T(W_1, \dots, W_n)(x_1, \dots, x_n) T(k^\mu)(y).
\end{aligned}$$

Then the second integral vanishes since k^μ is a conserved current. Partial integration in the first integral reveals according to the properties of H and f

$$\begin{aligned}
& \int d^4y f(y) \partial_\mu^y T(W_1, \dots, W_n, k^\mu)(x_1, \dots, x_n, y) = \\
& = [Q_c, T(W_1, \dots, W_n)(x_1, \dots, x_n)]_{\mp} = s_c T(W_1, \dots, W_n)(x_1, \dots, x_n). \quad (\text{B.6})
\end{aligned}$$

As the smearing of the right hand side of eqn. (N5) with f is trivial since $f = 1 \forall x_k$, we finally arrive at

$$\begin{aligned}
& s_c T(W_1, \dots, W_n)(x_1, \dots, x_n) = \\
& = \left(\sum_{k=1}^n g(W_k) \right) T(W_1, \dots, W_n)(x_1, \dots, x_n). \quad (\text{B.7})
\end{aligned}$$

The proof that the ghost number Ward identities have common solution with the other normalization conditions proceeds along the same lines as the proof of Dütsch and Fredenhagen [DF99] for the electric current. An important difference between the proofs is that for their proof it suffices to have eqn. (N4) for the basic generators, while it is here important to have it also for the higher generators since the ghost current k^μ contains also higher generators.

The proof is subdivided into two parts. At first we prove that it is possible to normalize $T(W_1 \cdots W_n)$ such that it satisfies eqn. (B.1). Then we prove the same statement for condition (N5). This seems to be a detour because we just saw that (B.1) is a consequence of (N5), but (B.1) will be needed in the proof of (N5).

Like all these proofs this one goes by induction, so we put forward the induction hypothesis that both (N5) and (B.1) hold for fewer arguments than n and for the sub monomials of the W_i , provided that at least one sub monomial is a proper one. Then the causal Wick expansion — eqn. (4.24) — tells us that eqn. (B.1) can only be violated by an unsuitable normalization of $\omega_0(T(W_1, \dots, W_n))$. Applying ω_0 to eqn. (B.1) and taking $\omega_0 \circ s_c = 0$ into account, we see that either $(\sum_{k=1}^n g(W_k)) = 0$ or $\omega_0(T(W_1, \dots, W_n)) = 0$. In the first case eqn. (B.1) is true for an arbitrary normalization of $\omega_0(T(W_1, \dots, W_n))$. In the second case validity of (B.1) in lower orders guarantees that $\omega_0(T(W_1, \dots, W_n))$ vanishes outside the diagonal but not necessarily on the entire M^n . Nevertheless it is always possible to extend a distribution that vanishes outside the diagonal by a distribution that vanishes everywhere, and such an extension is obviously compatible with all other normalization conditions. So it is always possible to find a normalization of $T(W_1, \dots, W_n)$ that is a solution of all normalization conditions including (B.1).

Now we come to the second part, the proof that normalizations can be found for

which the ghost number Ward identities **(N5)**

$$\begin{aligned} \partial_\mu^y T(W_1(x_1), \dots, W_n(x_n), k^\mu(y)) &= \\ &= \sum_{k=1}^n \delta(y - x_k) g(W_k) T(W_1(x_1), \dots, W_n(x_n)) \end{aligned} \quad (\text{B.8})$$

hold such that the normalization is also in accordance with **(N1)** - **(N4)**, provided none of the W_i is equal to k^μ or contains it as a sub monomial, and none of them contains generators $(u^a)^{(\alpha)}$ or $(\tilde{u}^a)^{(\alpha)}$ with $|\alpha| \geq 2$.

To this end we define a possible anomaly as

$$\begin{aligned} a(x_1, \dots, x_n, y) &= \partial_\mu^y T(W_1, \dots, W_n, k^\mu)(x_1, \dots, x_n, y) \\ &\quad - \sum_{k=1}^n \delta(y - x_k) g(W_k) T(W_1, \dots, W_n)(x_1, \dots, x_n) \end{aligned} \quad (\text{B.9})$$

and show that a normalization can be found — in agreement with eqns. **(N1)** - **(N4)** — such that the anomaly vanishes. Recalling our induction hypothesis we want to show this under the assumption that all these anomalies vanish for the time ordered products of fewer arguments than n and in all equations that involve the sub monomials of the W_i . The proof will be divided into three steps.

Step 1: At first we commute the anomaly with the basic fields $\varphi_i(x)$ in order to find that this commutator vanishes. Thereby we make repeated use of condition **(N3)** and the fact that according to our induction hypothesis eqn. **(N5)** is already established for the lower orders and for the sub monomials. The result of that calculation is

$$\begin{aligned} \left[a(x_1, \dots, x_n, y), \varphi_i(z) \right]_{\mp} &= \\ &= ig(\varphi_i) \sum_{k=1}^n \sum_j \Delta_{ij}(x_k - z) \delta(y - x_k) T\left(W_1, \dots, \frac{\partial W_k}{\partial \varphi_j}, \dots, W_n\right) \\ &\quad + i \sum_j (\partial_\mu^y \Delta_{ij}(y - z)) T\left(W_1, \dots, W_n, \frac{\partial k^\mu}{\partial \varphi_j}\right) \\ &\quad + i \sum_j \Delta_{ij}(y - z) \partial_\mu^y T\left(W_1, \dots, W_n, \frac{\partial k^\mu}{\partial \varphi_j}\right), \end{aligned} \quad (\text{B.10})$$

where we have omitted the spacetime arguments of the time ordered products because the expressions would not fit into the line otherwise. We will do this throughout this proof. It should not cause confusion since it is already clear from the arguments of the time ordered products which the spacetime arguments are.

To show that the expression above vanishes we distinguish three cases:

Case 1: $\varphi_i(z) \neq u^a(z), \tilde{u}^a(z)$. In this case both $\frac{\partial k^\mu}{\partial \varphi_i} = 0$ and $g(\varphi_i) = 0$, so the commutator vanishes immediately.

Case 2: $\varphi_i(z) = u^a(z)$. At first we note that $g(u^a) = 1$. Furthermore we have $k^\mu = i(\tilde{u}_a)^{(1,\mu)} u^a - i\tilde{u}_a(u^a)^{(1,\mu)}$, so we get in particular $\frac{\partial k^\mu}{\partial u_a} = -i(u^a)^{(1,\mu)}$ and $\frac{\partial k^\mu}{\partial (\tilde{u}_a)^{(1,\nu)}} = i\delta_\nu^\mu u^a$. Taking this and the definition of the commutator function Δ_{ij}

into account, we get for the last two lines in (B.10)

$$\begin{aligned}
& (\partial_\mu^y D(y-z)) T(W_1, \dots, W_n, (u^a)^{(1,\mu)}) \\
& + D(y-z) \partial_\mu^y T(W_1, \dots, W_n, (u^a)^{(1,\mu)}) \\
& - (\partial_\mu^y D(y-z)) \partial_y^\mu T(W_1, \dots, W_n, u^a).
\end{aligned} \tag{B.11}$$

According to (N4) the expression above transforms into

$$\begin{aligned}
& (\partial_y^\mu D(y-z)) \left[-i C_{u,1} \sum_{k=1}^n \delta(y-x_k) T\left(W_1, \dots, \frac{\partial W_k}{\partial(\tilde{u}^a)^{(1,\mu)}}, \dots, W_n\right) \right] \\
& + D(y-z) \left[+i C_{u,1} \sum_{k=1}^n \delta(y-x_k) T\left(W_1, \dots, \frac{\partial W_k}{\partial \tilde{u}^a}, \dots, W_n\right) \right. \\
& \quad - (1 + C_{u,1}) \square T(W_1, \dots, W_n, u^a) \\
& \quad \left. - i \frac{C_{u,1}}{C_{u,2}} \partial_y^\alpha \partial_y^\beta \left[\sum_{k=1}^n \delta(x_k - y) T\left(W_1, \dots, \frac{\partial W_k}{\partial(\tilde{u}^a)^{2,\alpha\beta}}, \dots, W_n\right) \right] + \dots \right]
\end{aligned} \tag{B.12}$$

Since we required that the W_i do not contain generators $(u^a)^{(\alpha)}$ with $|\alpha| \geq 2$, the last line and the following terms containing derivatives w.r.t. higher generators on the W_i , indicated by the dots, vanish. Then comparing the remaining expression with the first line in (B.10) reveals that these expressions cancel each other if and only if $C_{u,1} = -1$. So the choice $C_{u,1} = -1$ is a necessary (and, as it will turn out, sufficient) condition for eqn. (N5) to hold.

Case 3: $\varphi_i(z) = \tilde{u}^a(z)$. The calculation for this case is completely analogous the one before and reveals $C_{u,1} = -1$ as a necessary condition for the commutator to vanish, too.

So with $C_u = -1$ the commutator of the anomaly with every free field vanishes. Consequently $a(x_1, \dots, x_n, y)$, smeared with an arbitrary test function, is a \mathbb{C} -number distribution.

Step 2: We already know that time ordered products with at least one generator as an argument are completely determined by the time ordered products in lower orders and those for the sub monomials. We will now examine whether this normalization is compatible with (N5).

Since the anomaly can at most be a \mathbb{C} -number distribution, it is sufficient to calculate its \mathbb{C} -number part $\omega_0(a(x_1, \dots, x_n, y))$. So we want to prove that

$$\begin{aligned}
& \partial_\mu^y \omega_0 \left(T(W_1, \dots, W_n, \varphi_i, k^\mu) \right) = \\
& \left(\sum_{k=1}^n \delta(y-x_k) g(W_k) + \delta(y-z) g(\varphi_i) \right) \omega_0 \left(T(W_1, \dots, W_n, \varphi_i) \right).
\end{aligned} \tag{B.13}$$

With a repeated use of eqn. (4.29) this can be transformed into

$$\begin{aligned}
& i g(\varphi_i) \sum_{k=1}^n \sum_j \Delta_{ij}^F(z-x_k) \delta(y-x_k) \omega_0 \left(T \left(W_1, \dots, \frac{\partial W_k}{\partial \varphi_j}, \dots, W_n \right) \right) \\
& + i \sum_j (\partial_\mu^y \Delta_{ij}^F(z-y)) \omega_0 \left(T \left(W_1, \dots, W_n, \frac{\partial k^\mu}{\partial \varphi_j} \right) \right) \\
& + i \sum_j \Delta_{ij}^F(z-y) \partial_\mu^y \omega_0 \left(T \left(W_1, \dots, W_n, \frac{\partial k^\mu}{\partial \varphi_j} \right) \right) \\
& = g(\varphi_i) \delta(y-z) \left(i \sum_{k=1}^n \sum_j \Delta_{ij}^F(z-x_k) \omega_0 \left(T \left(W_1, \dots, \frac{\partial W_k}{\partial \varphi_j}, \dots, W_n \right) \right) \right). \tag{B.14}
\end{aligned}$$

Again we can distinguish different cases here.

In the first case, $\varphi_i \neq (u^a)^{(\alpha)}, (\tilde{u}^a)^{(\alpha)}$, we have again both $\frac{\partial k^\mu}{\partial \varphi_i} = 0$ and $g(\varphi_i) = 0$, so the equation holds automatically. The cases $\varphi_i = (u^a)^{(\alpha)}$ or $\varphi_i = (\tilde{u}^a)^{(\alpha)}$ with $|\alpha| \geq 2$ cannot occur because they were explicitly excluded. So there remain four cases where we have to prove that the equation above is indeed valid: $\varphi_i = (u^a), (u^a)^{(1,\mu)}, (\tilde{u}^a)$ and $(\tilde{u}^a)^{(1,\mu)}$. For simplicity we will treat only $\varphi_i = (u^a)$, the calculation for the other cases is analogous.

Remembering $g(u^a) = 1$, $\frac{\partial k^\mu}{\partial \tilde{u}^a} = -i(u^a)^{(1,\mu)}$ and $\frac{\partial k^\mu}{\partial (\tilde{u}^a)^{(1,\mu)}} = i\delta_\nu^\mu(u^a)$ from the first step, we see that the sum of the second and third line on the left hand side of equation (B.14) give, where eqn. (N4) has been used,

$$\begin{aligned}
& (\partial_y^\mu D^F(z-y)) \left[i \sum_{k=1}^n \delta(y-x_k) \omega_0 \left(T \left(W_1, \dots, \frac{\partial W_k}{\partial (\tilde{u}^a)^{(1,\mu)}}, \dots, W_n \right) \right) \right] \\
& + D^F(z-y) \left[i \sum_{k=1}^n \delta(y-x_k) \omega_0 \left(T \left(W_1, \dots, \frac{\partial W_k}{\partial (\tilde{u}^a)}, \dots, W_n \right) \right) \right] \\
& - \delta(z-y) \left[i \sum_{k=1}^n D^F(y-x_k) T \left(W_1, \dots, \frac{\partial W_k}{\partial (\tilde{u}^a)}, \dots, W_n \right) \right. \\
& \quad \left. i \sum_{k=1}^n (\partial_y^\mu D^F(y-x_k)) T \left(W_1, \dots, \frac{\partial W_k}{\partial (\tilde{u}^a)^{(1,\mu)}}, \dots, W_n \right) \right]. \tag{B.15}
\end{aligned}$$

Comparing this with the other lines in eqn. (B.14), we see that the last two lines cancel the right hand side of that equation while the first two lines cancel the first line on the the left hand side. So the equation is indeed satisfied. As we already remarked, it can be proven by an analogous calculation that this is also true if $\varphi_i = (u^a)^{(1,\mu)}, (\tilde{u}^a)$ or $(\tilde{u}^a)^{(1,\mu)}$. With this we have proven that the time ordered products with at least one generator among their arguments satisfy condition (N5) automatically.

Step 3: We know up to now that eqn. (N5) can only be violated by T -products that have no generator among their arguments, and this violation can be at most a \mathbb{C} -number. In addition we know that the anomaly must be local because of causal factorization and validity of (N5) in lower orders, so it can be written as

$$a(x_1, \dots, x_n, y) = \omega_0(a(x_1, \dots, x_n, y)) = P(\partial) \delta(y-x_1) \cdots \delta(y-x_n) \tag{B.16}$$

for some polynomial of spacetime derivatives $P(\partial)$. To show that such an anomaly can always be removed we notice that

$$0 = \int d^4y f(y) a(x_1, \dots, x_n, y) = \int d^4y a(x_1, \dots, x_n, y) \quad (\text{B.17})$$

where f is a test function like in the proof of eqn. (4.32). The first identity is an immediate consequence of that equation. This is the point in the proof of **(N5)** where it is necessary to know in advance that (B.1) holds. The second identity is true since $f = 1$ in a domain around each x_k .

Let us consider the Fourier transformation of the anomaly,

$$\begin{aligned} \hat{a}(x_1, \dots, x_n, y) &= (2\pi)^n \int d^4x_1 \cdots d^4x_n a(x_1, \dots, x_n, y) e^{i(p_1x_1 + \cdots + p_nx_n)} \\ &= (2\pi)^n P(-ip_1, \dots, -ip_n) e^{i(p_1 + \cdots + p_n)y}. \end{aligned} \quad (\text{B.18})$$

For the second identity we have adopted eqn. (B.16) for the anomaly. Inserting (B.18) back into eqn. (B.17), we find that the polynomial $P(-ip_1, \dots, -ip_n)$ vanishes on the hyperplane $p_1 + \cdots + p_n = 0$:

$$P(-ip_1, \dots, -ip_n) \delta(p_1 + \cdots + p_n) = 0. \quad (\text{B.19})$$

Now we define $\tilde{P}(q, p_1, \dots, p_{n-1}) \stackrel{\text{def}}{=} P(-ip_1, \dots, -ip_n)$ with $q \stackrel{\text{def}}{=} p_1 + \cdots + p_n$ and consider its Taylor expansion around the origin:

$$\tilde{P}(q, p_1, \dots, p_{n-1}) = \sum_{k=1}^{\text{degree}\tilde{P}} \sum_{|\alpha|+|\beta|=k} \frac{q^\alpha p^\beta}{\alpha! \beta!} \left(\frac{\partial^{|\alpha|} \partial^{|\beta|}}{\partial q^\alpha \partial p^\beta} \tilde{P} \right) (0) \quad (\text{B.20})$$

where $p \stackrel{\text{def}}{=} (p_1, \dots, p_{n-1})$. So the derivatives $\frac{\partial^{|\alpha|}}{\partial q^\alpha}$ describe a variation orthogonal to the hyperplane $p_1 + \cdots + p_n = 0$, the derivatives $\frac{\partial^{|\beta|}}{\partial p^\beta}$ a variation within it. Since \tilde{P} vanishes throughout the entire plane, terms with $|\alpha| = 0$ must vanish. Therefore the Taylor expansion can be rewritten as

$$\tilde{P}(q, p_1, \dots, p_{n-1}) = q \cdot \sum_{k=0}^{\text{degree}\tilde{P}-1} \sum_{|\alpha|+|\beta|=k} \frac{q^\alpha p^\beta}{\alpha! \beta!} \left(\frac{\partial^{|\alpha|} \partial^{|\beta|}}{\partial q^\alpha \partial p^\beta} \tilde{P}_1^\mu \right) (0) \quad (\text{B.21})$$

with a new polynomial \tilde{P}_1^μ . Reversing the Fourier transformation we find

$$P(\partial) = \left(\sum_{i=1}^n \partial_\mu^i \right) P_1^\mu(\partial) \quad (\text{B.22})$$

where the polynomial P_1^μ is the Fourier transform of \tilde{P}_1^μ . With this expression we can write the anomaly as

$$a(x_1, \dots, x_n, y) = -\partial_\mu^y (n \cdot P_1^\mu \delta(x_1 - y) \cdots \delta(x_n - y)). \quad (\text{B.23})$$

So the anomaly can be removed by addition of $n \cdot P_1^\mu \delta(x_1 - y) \cdots \delta(x_n - y)$ to the the previous normalization of $T(W_1, \dots, W_n, k^\mu)(x_1, \dots, x_n, y)$. This is obviously a valid normalization and so the desired normalization has been found.

The question remains why we had excluded polynomials with k^μ as a sub polynomial. The reason is that we can assure that a normalization with the desired properties exists, but we cannot assure that time ordered products like

$$T(k^\mu, k^\nu, W_1, \dots, W_n)(y, z, x_1, \dots, x_n) \quad (\text{B.24})$$

are symmetric under simultaneous exchange of μ, ν and y, z as they must. There is indeed a counterexample for the Ward identities of the axial current $j_A^\mu =: \bar{\psi} \gamma^\mu \gamma^5 \psi :$, where it is not possible to find a normalization of $T(j_A^\mu, j_A^\mu, j_A^\mu)(x, y, z)$ with the required symmetries. Excluding the respective polynomials from the allowed arguments makes sure that this situation does not occur.

From the proof above it is clear that the normalization we have found is compatible both with **(N3)** and **(N4)**. But we can also immediately see that **(N5)** respects Poincaré transformation properties and therefore **(N1)**. Taking the adjoint of **(N5)** finally reveals that it complies also with **(N2)** and therefore eventually with all other normalization conditions.

B.2. Relation between **(N6)** and generalized operator gauge invariance.

It is always possible to define an operator valued distribution $T(\mathcal{L}_{i_1}, \dots, \mathcal{L}_{i_n}, j^\mu)$ by

$$\begin{aligned}
T(\mathcal{L}_{i_1}, \dots, \mathcal{L}_{i_n}, j^\mu)(x_1, \dots, x_n, y) &\stackrel{\text{def}}{=} \\
&\stackrel{\text{def}}{=} -s_0 T(\mathcal{L}_{i_1}, \dots, \mathcal{L}_{i_n}, k^\mu)(x_1, \dots, x_n, y) \\
&\quad + i \sum_{m=1}^n \partial_\nu^m T(\mathcal{L}_{i_1}, \dots, \mathcal{L}_{i_{m+1}}^\nu, \dots, \mathcal{L}_{i_n}, k^\mu)(x_1, \dots, x_n, y) \\
&\quad - i \sum_{m=1}^n \delta(x_m - y) T(\mathcal{L}_{i_1}, \dots, \mathcal{M}_{i_{m+1}}^\mu, \dots, \mathcal{L}_{i_n})(x_1, \dots, x_n) \\
&\quad + i \sum_{m=1}^n \delta(x_m - y) \cdot i_m \cdot T(\mathcal{L}_{i_1}, \dots, \mathcal{L}_{i_{m+1}}^\mu, \dots, \mathcal{L}_{i_n})(x_1, \dots, x_n).
\end{aligned} \tag{B.25}$$

$T(\mathcal{L}_{i_1}, \dots, \mathcal{L}_{i_n}, j^\mu)$ is at this point only a name for that distribution, we must still prove that it is indeed an extension of ${}^0T(\mathcal{L}_{i_1}, \dots, \mathcal{L}_{i_n}, j^\mu)$. Before we do that, we point out that it implicates **(N6)** almost immediately. Of course the time ordered products on the right hand side must satisfy eqn. **(N5)**. Taking the derivative w.r.t. the y coordinate, we find with **(N5)**

$$\begin{aligned}
\partial_\mu^y T(\mathcal{L}_{i_1}, \dots, \mathcal{L}_{i_n}, j^\mu)(x_1, \dots, x_n, y) &= \\
&= - \left(\sum_{m=1}^n \delta(x_m - y) \cdot i_m \right) s_0 T(\mathcal{L}_{i_1}, \dots, \mathcal{L}_{i_n})(x_1, \dots, x_n) \\
&\quad + i \sum_{l=1}^n \partial_\nu^l \left[T(\mathcal{L}_{i_1}, \dots, \mathcal{L}_{i_{l+1}}^\nu, \dots, \mathcal{L}_{i_n})(x_1, \dots, x_n) \times \right. \\
&\quad \quad \quad \left. \times \left(\sum_{m=1}^n \delta(x_m - y) \cdot i_m + \delta(x_l - y) \right) \right] \\
&\quad + i \sum_{m=1}^n \left(\partial_\nu^m \delta(x_m - y) \right) T(\mathcal{L}_{i_1}, \dots, \mathcal{M}_{i_{m+1}}^\nu, \dots, \mathcal{L}_{i_n})(x_1, \dots, x_n) \\
&\quad - i \sum_{m=1}^n \left(\partial_\nu^m \delta(x_m - y) \cdot i_m \right) T(\mathcal{L}_{i_1}, \dots, \mathcal{L}_{i_{m+1}}^\nu, \dots, \mathcal{L}_{i_n})(x_1, \dots, x_n).
\end{aligned} \tag{B.26}$$

Smearing out this equation with a test function f like the one defined following eqn. (B.2) gives eqn. (B.1), the calculation is the same as at the beginning of the last section. Inserting this result into eqn. (B.26) we get immediately eqn. (N6). Eqn. (B.25) is obviously a well posed definition since all operations involved in it are well defined — in-particular the time ordered product in the last line contains no vertex at y , so the product with the delta distribution is a tensor product.

The crucial question is whether the operator valued distribution has the correct causal factorization outside the diagonal Diag_{n+1} . Only then it is really a time ordered product of its arguments as the notation suggests. Basically we must do the same construction as in the respective point for (N4), see section (A.2). We give here only a simplified version of this proof where the essential point may be more easily understood. The detailed version can easily be derived from this sketch. Suppose the points x_1, \dots, x_n are in a relative position such that

$$\emptyset \neq I = \{x_1, \dots, x_k\} \gtrsim \{x_{k+1}, \dots, x_n, y\}. \quad (\text{B.27})$$

This is the situation we encounter in eqn. (A.17) in the first sum — if $I = \{x_1, \dots, x_k\} \lesssim \{x_{k+1}, \dots, x_n, y\}$, corresponding to the second sum there, the argument works as well. Then

$$T(\mathcal{L}_{i_1}, \dots, \mathcal{L}_{i_n}, j^\mu) = T(\mathcal{L}_{i_1}, \dots, \mathcal{L}_{i_k}) T(\mathcal{L}_{i_{k+1}}, \dots, \mathcal{L}_{i_n}, j^\mu) \quad (\text{B.28})$$

where we omitted the spacetime indices for simplicity. Eqn. (B.25) is valid for $T(\mathcal{L}_{i_{k+1}}, \dots, \mathcal{L}_{i_n}, j^\mu)$ since we assumed that eqn. (B.25) holds already for time ordered products with fewer arguments. Together with eqn. (B.28) this gives the following expression

$$\begin{aligned} T(\mathcal{L}_{i_1}, \dots, \mathcal{L}_{i_n}, j^\mu) &= \\ &= -s_0 [T(\mathcal{L}_{i_1}, \dots, \mathcal{L}_{i_k}) T(\mathcal{L}_{i_{k+1}}, \dots, \mathcal{L}_{i_n}, k^\mu)] \\ &\quad + [s_0 T(\mathcal{L}_{i_1}, \dots, \mathcal{L}_{i_k}) T(\mathcal{L}_{i_{k+1}}, \dots, \mathcal{L}_{i_n}, k^\mu)] \\ &\quad + \sum_{l=k+1}^n \partial_\nu^l [T(\mathcal{L}_{i_1}, \dots, \mathcal{L}_{i_k}) T(\mathcal{L}_{i_{k+1}}, \dots, \mathcal{L}_{i_l+1}^\nu, \dots, \mathcal{L}_{i_n}, k^\mu)] \\ &\quad - i \sum_{l=k+1}^n \delta(x_l - y) [T(\mathcal{L}_{i_1}, \dots, \mathcal{L}_{i_k}) T(\mathcal{L}_{i_{k+1}}, \dots, \mathcal{M}_{i_l+1}^\mu, \dots, \mathcal{L}_{i_n})] \end{aligned} \quad (\text{B.29})$$

where we have omitted spacetime arguments for simplicity. For $s_0 T(\mathcal{L}_{i_1}, \dots, \mathcal{L}_{i_k})$ we may use the generalized operator gauge invariance (4.35) in lower orders as long as $k \neq n$. Unfortunately also the case $k = n$ occurs if all the x_i coincide and only y is separated from them. This is the only case where we must know in advance that (4.35) holds. If this would not be true then our definition (B.25) would be a well defined operator valued distribution, but not an extension of $T^0(\mathcal{L}_{i_1}, \dots, \mathcal{L}_{i_n}, j^\mu)$ to the diagonal — this means that it could differ from the T^0 -product even outside the diagonal. Hence we need to assume that (4.35) is valid also for $k = n$. Furthermore we may add in the last sum the terms with $l = 1, \dots, k$ since the delta distributions vanish because y and the x_1, \dots, x_k may never coincide. Recombining the products of T -products into a single T -product according to eqn. (B.28) one gets immediately (B.25). As already remarked the calculation comes to the same result if $\emptyset \neq I = \{x_1, \dots, x_k\} \lesssim \{x_{k+1}, \dots, x_n, y\}$. So (B.25) is a well defined operator valued distribution that agrees — as long as (4.35) is valid — with $T^0(\mathcal{L}_{i_1}, \dots, \mathcal{L}_{i_n}, j^\mu)$ if smeared with a test function that vanishes with all its derivatives on the diagonal,

so it is an extension of that T^0 -product to the diagonal and therefore a possible normalization of $T(\mathcal{L}_{i_1}, \dots, \mathcal{L}_{i_n}, j^\mu)$.

So we have just proven that the conditions **(N6)** and (4.35) are equivalent.

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