Symmetry Groups in Physics: Problems

Problem 22 — Point groups in two dimensions
In two dimensions a point group is a finite subgroup of $O(2)$. Classify all point groups in two dimensions!

Problem 23 — Lattice
Let $\{v_1, ..., v_n\}$ be a basis of the real linear space $\mathbb{R}^n$. The set

$$\left\{ \sum_{k=1}^{n} m_k v_k \,\middle|\, m_1, ..., m_n \in \mathbb{Z} \right\}$$

is the "lattice" generated by the basis.

a) Show that any lattice is a subgroup of $(\mathbb{R}^n, +)$!

b) Show that any lattice in $\mathbb{R}^n$ is isomorphic to the n-fold direct product of $(\mathbb{Z}, +)$ with itself.

Problem 24 — Helmholtz equation
Consider the Helmholtz equation

$$\Delta f(x) + k^2 f(x) = 0 \qquad (x \in \mathbb{R}^3,\ k = \omega/c)\ .$$

The operator $\Delta + k^2$ is isotropic and homogeneous ($\Delta = \nabla^2$). Hence, with f also $D(g)f$ is a solution, where $g = (a, R) \in E(3)$ is an isometry and where

$$[D(g)f](x) := f(g^{-1}x) = f(R^{-1}(x - a))\ .$$

Show that D is an infinite-dimensional representation of the Euclidean group!