Symmetry Groups in Physics: Problems

Problem 19 — Z_4 as a semidirect product?
We have $Z_2 \times Z_2 = K_4$. Can one get the other group of order 4, namely Z_4, by some suitably defined semidirect product of Z_2 with Z_2?

Problem 20 — Euclidean group
Argue that a rotation $R \in O(n)$ around a point a (with $a \neq 0$ in general) is given by the following element of the Euklidean group:

$$(a, \mathbf{1}) * (0, R) * (-a, \mathbf{1}) \in E(n) \;!$$

Prove that the rotations around a point a

$$O_a(n) = \{(a - Ra, R) \,|\, R \in O(n)\} < E(n)$$

form a subgroup!

Show that $O_a(n) \cong O(n)$!

Demonstrate that a rotation R_2 around a_2 followed by a rotation R_1 around a_1 is the same as a rotation $R_1 R_2$ around a_2 followed by a translation $(a, \mathbf{1})$! Compute a!

Problem 21 — Galilei group
The Galilei group G consists of time translations, spatial translations, rotations and of Galilei transformations between inertial frames of reference.

a) How many and which independent parameters characterize an element $g \in G$?

b) How does G act on $\mathbb{R}^4$?

c) Consider the 5-tuple $(x, t, 1)^T \in \mathbb{R}^5$, and show that $G < GL(5, \mathbb{R})$!

d) Derive the explicit rule for the multiplication of two elements g, g'!

e) Identify normal subgroups!

f) Show that G is the semi-direct product of $\mathbb{R}^4 = \mathbb{R} \times \mathbb{R}^3$ (temporal and spatial translations) with $E(3) = \mathbb{R}^3 \rtimes O(3)$,

$$G = (\mathbb{R} \times \mathbb{R}^3) \rtimes (\mathbb{R}^3 \rtimes O(3))$$

where the automorphism is

$$\Phi_{(R,v)}(\tau, a) = (\tau, Ra + v\tau) \quad !$$