Symmetry Groups in Physics: Problems

Problem 15 — Symplectic matrices
For arbitrary integer n, the set of symplectic matrices is defined as

$$Sp(2n,\mathbb{R}) = \{A \in GL(2n,\mathbb{R}) \,|\, A^T J A = J\}$$

where $J \in GL(2n,\mathbb{R})$ is a fixed but arbitrary (!) skew symmetric $2n \times 2n$ matrix ($J_{ij} = -J_{ji}$).

a) Show that $Sp(2n,\mathbb{R})$ with the usual matrix multiplication forms a group!

b) Consider iJ to show that the eigenvalues of J are purely imaginary and come in pairs, $i\lambda_j, -i\lambda_j$ (λ_j real)!

c) With the same idea, show that there is a real orthogonal matrix O such that

$$O^T J O = \text{diag}\left(\begin{pmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & \lambda_n \\ -\lambda_n & 0 \end{pmatrix}\right) \equiv D\ !$$

Hint: Consider $y_j \equiv (x_j + x_j^*)/\sqrt{2}$ and $z_j \equiv -i(x_j - x_j^*)/\sqrt{2}$ where x_j is the eigenvector of J corresponding to the eigenvalue $i\lambda_j$.

d) The symplectic group is unique! Show that

$$Sp_J(2n,\mathbb{R}) = \{A \in GL(2n,\mathbb{R}) \,|\, A^T J A = J\}$$

and

$$Sp_{J'}(2n,\mathbb{R}) = \{A \in GL(2n,\mathbb{R}) \,|\, A^T J' A = J'\}$$

are isomorphic for arbitrary real skew-symmetric J and J'!

e) Show $\det A = +1$ for $A \in Sp(2n,\mathbb{R})$!

Hint: Show that if λ_0 is an eigenvalue of A with multiplicity k, then $1/\lambda_0$ must be an eigenvalue of A with the same multiplicity!