Symmetry Groups in Physics: Problems

Problem 10 — Left and right translations
a) Show that the map $G \to S(G)$, $a \mapsto l_a$ is a group homomorphism! Here, l_a is the left translation, i.e., $l_a(x) = ax$.

b) Show that the map $G \to S(G)$, $a \mapsto r_a$ is a group homomorphism only if G is abelian! Here, r_a is the left translation, i.e., $r_a(x) = xa$.

c) Show that with the modified right translation $x \mapsto xa^{-1}$ one can construct a group homomorphism $G \to S(G)$ for any G!

Problem 11 — Conjugation of normal subgroups
Prove the following: If N is a normal subgroup of G then $\varphi_a(N)$ is a normal subgroup of G for all $a \in G$! Here $\varphi_a : G \to G$ with $\varphi_a(x) = axa^{-1}$ is the conjugation.

Problem 12 — Subgroups with index 2
Show that any subgroup $H < G$ of a group G with index $|G : H| = 2$ is a normal subgroup.

Problem 13 — Commutator group
For a group G, the commutator group $[G, G]$ is defined as the group which is generated by the commutators

$$\{[a,b] = aba^{-1}b^{-1} \mid a, b \in G\} ,$$

i.e.

$$[G,G] = \langle\{[a,b] = aba^{-1}b^{-1} \mid a, b \in G\}\rangle .$$

Show that

a) $[G, G] < G$.

b) $[G, G]$ is a normal subgroup. Hint: consider the conjugation!

c) The factor group $G/[G, G]$ is abelian.

Problem 14 — Group center
Determine the image of the center of a group under an automorphism!