Symmetry Groups in Physics: Problems

Problem 8 — Classification of all cyclic groups
Let $G = (g) = \{g^k \mid k \in \mathbb{Z}\}$ be an arbitrary cyclic group.

Assume $|G| = \infty$ and consider the map

$$\phi : \mathbb{Z} \to G \ , \quad k \mapsto g^k \ .$$

Show that ϕ is an isomorphism!

Assume $|G| < \infty$. Define m as the smallest positive integer with $g^m = e$, write $G = \{e, g, g^2, \dots g^{m-1}\}$ and consider the map

$$\phi : \mathbb{Z}/m\mathbb{Z} \to G \ , \quad k + m\mathbb{Z} \mapsto g^k \ .$$

Show that ϕ is an isomorphism!

Prove the following theorem:
Let G be a cyclic group. Then $G \cong \mathbb{Z}$ if $|G| = \infty$ and $G \cong Z_m$ if $|G| = m < \infty$.

Problem 9 — Canonical transformations
Consider a classical system with n degrees of freedom, the dynamics of which is described by a Hamilton function $H(x) = H(q_1, ..., q_n, p_1, ..., p_n)$ where the $2n$-dimensional vector $x = (q_1, ..., q_n, p_1, ..., p_n)$ is the phase, i.e., the state of the system. Show the following propositions:

a) The equation of motion for the system's state can be written as:

$$\dot{x} = J \nabla H(x) \ ,$$

where

$$J = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}$$

where and $\mathbf{1}$ is the $n \times n$ unity matrix.

b) The Poisson bracket of two observables $A(x)$ and $B(x)$ takes the form

$$\{A, B\} = (\nabla A, J \nabla B)$$

where $(..., ...)$ is the usual Euclidean scalar product.

c) An arbitrary (smooth) transformation $y = y(x)$ in the $2n$-dimensional phase space leaves the form of the equation of motion invariant, i.e.,

$$\dot{y} = J \nabla_y H(x(y)) \ ,$$

if and only if

$$SJS^T = J$$

where S is the Jacobian of the transformation, i.e,

$$S_{ij} = \frac{\partial y_i}{\partial x_j}$$

and $i, j = 1, ..., 2n$.

d) A transformation $y = y(x)$ is canonical, iff

$$\{y_i, y_j\} = J_{ij}$$

for all $i, j = 1, ..., 2n$.