Symmetry Groups in Physics: Problems

Problem 5 — Sets of left and right cosets
Let H be a subgroup of the finite group G and consider the set of the left cosets G/H and the set of right cosets $H\backslash G$.

Show that the map

$$\tau : G/H \to H\backslash G\ , \qquad \tau(aH) = Ha^{-1} \quad \text{with } a \in G$$

is bijective!

Problem 6 — Quaternion group
$GL(n, \mathbb{C})$ is the set of all invertible $n \times n$ matrices over the field of complex numbers. Show that $GL(n, \mathbb{C})$ with the usual matrix multiplication forms a group!

Consider the following elements of $GL(2, \mathbb{C})$:

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad I = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad K = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} .$$

The quaternion group is the subgroup of $GL(n, \mathbb{C})$ generated by these elements:

$$Q_8 = \langle E, I, J, K \mid \varnothing \rangle .$$

List all elements of Q_8! Is Q_8 abelian?

Draw the cycle graph!

Show that there are three cyclic subgroups of order 4 and one cyclic subgroup of order 2!

A subgroup N of a group G is a normal subgroup if $gN = Ng$ for all $g \in G$. Which of the cyclic subgroups are normal?

Is Q_8 a normal subgroup of $GL(n, \mathbb{C})$?

What is the center of Q_8?

Determine Cen(I) and Cl(I)!

Let $\mathbb{H}$ be the set of all real linear combinations of E, I, J, K. Show that $\mathbb{H}$ with the usual matrix multiplication and addition forms a so-called division ring or skew field, i.e. all field axioms hold except for the commutative property of the multiplication!

For $z \in \mathbb{H}$, compute z^{-1} explicitly! (Hint: How is this done for complex numbers?)

Problem 7 — Orbits

Let X be a non-empty set and G a group which (left) *operates* on X, i.e. there is a map $G \times X \to X$ with $(a, x) \mapsto a \cdot x = ax$ with the properties

(i) $a(bx) = (ab)x$ for all $a, b \in G$ and all $x \in X$.

(ii) $ex = x$ for all $x \in X$ (e is the neutral element in G).

For a given $x \in X$, we call $Gx = \{ax \mid a \in G\}$ the "orbit" of x under the operation of G. Show that:

a) $x \sim y :\Leftrightarrow \exists a \in G$: $x = ay$ defines an equivalence relation for the elements of X! The orbit of x is the equivalence class of x.

b) X is the disjoint union of the orbits of certain $x \in X$!

c) Find 10 examples for groups operating on sets!