

## Symmetry Groups in Physics: Problems

### Problem 33 — Torus

The “Closed Subgroup Theorem” (der Satz von Cartan) states that a subgroup  $H$  of a Lie group  $G$  is a Lie subgroup if and only if  $H$  is closed (i.e.  $H = H \cup \partial H$ ).

Furthermore, a normal subgroup  $H$  of a Lie group  $G$  is a normal Lie subgroup if and only if  $H$  is closed (i.e.  $H = H \cup \partial H$ ).

a) Consider, as an example, the torus:

$$G = T^2 = \left\{ \begin{pmatrix} e^{i\varphi_1} & 0 \\ 0 & e^{i\varphi_2} \end{pmatrix} \mid \varphi_1, \varphi_2 \in \mathbb{R} \right\}$$

and show that  $G$  is a Lie group!

b) Compute the structure constants of the corresponding Lie algebra!

c) Show that

$$H = \left\{ \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{i\varphi} \end{pmatrix} \mid \varphi \in \mathbb{R} \right\}$$

is a normal Lie subgroup of  $G$ !

d) Show that

$$H' = \left\{ \begin{pmatrix} e^{2\pi i\varphi} & 0 \\ 0 & e^{2\pi i r\varphi} \end{pmatrix} \mid \varphi \in \mathbb{R} \right\} \quad \text{with irrational } r$$

is not a Lie subgroup of  $G$ !