Symmetry Groups in Physics: Problems

Problem 29 — Lie algebra from structure constants
Assume that n^3 real numbers f_{ijk} (with $i, j, k = 1, ..., n$) are given and satisfy the properties

$$\begin{aligned} f_{ijk} &= -f_{jik} \\ 0 &= \sum_l (f_{jkl} f_{ilm} + f_{ijl} f_{klm} + f_{kil} f_{jlm}) \ . \end{aligned}$$

Furthermore, there is an n-dimensional linear space g with basis $\{T_1, ...T_n\}$.

How can one define a Lie algebra? Verify the according postulates!

Problem 30 — Classical analogue of a Lie algebra
Assume that a Lie algebra is given by the generators $T_1, ..., T_n$ and the commutation relations

$$[T_i, T_j] = \sum_k f_{ijk} T_k \ ,$$

where f_{ijk} are the structure constants. T_i are Hermitian operators in a Hilbert space.

To find the *classical* counterpart of the corresponding quantum system, we define the Lie-Poisson bracket for arbitrary functions $A = A(X)$ and $B = B(X)$ of $X = (X_1, ..., X_n) \in \mathbb{R}^n$ as

$$\{A, B\} = \sum_{ijk} \frac{\partial A}{\partial X_i} \frac{\partial B}{\partial X_j} X_k f_{ijk} \ .$$

Verify that the known general properties of a Poisson bracket are recovered: antisymmetry, bilinearity and the Jacobi identity! Show that this definition satisfies

$$\{X_i, X_j\} = \sum_k f_{ijk} X_k \quad !$$

Apply this idea to $su(2)$ (complex and traceless 2×2 matrices) to get the classical analog of the spin-$1/2$ algebra!

What form do the classical canonical equations of motion

$$\dot{A} = \{A, H\}$$

take in this case if one considers for $A = X_i$ and a Hamiltonian of the form $H = H(X) = -\sum_i B_i X_i$, where B_i $(i = 1, 2, 3)$ is some external field?

Problem 31 — Orthogonality of irreducible representations
Let $D : G \to \mathsf{GL}(V)$ and $D' : G \to \mathsf{GL}(V')$ be two irreducible representations of a finite group G in linear spaces V and V'.

Consider an arbitrary linear map $B : V \to V'$, and define a "G-symmetrized" linear map $A : V \to V'$:

$$A = \frac{1}{|G|} \sum_{a \in G} D'(a^{-1}) B D(a)$$

and show that

(i) $A = 0$ if D and D' are inequivalent.

(ii) $A = \text{tr}B/\dim V$ if $D = D'$ (and $V = V'$)!

Choose bases of V and V' and consider the respective matrix representations of D and D'. Show that

(iii) $\frac{1}{|G|} \sum_{a \in G} D'_{i'j'}(a^{-1}) D_{ji}(a) = 0$ if D, D' are inequivalent.

(iv) $\frac{\dim V}{|G|} \sum_{a \in G} D_{i'j'}(a^{-1}) D_{ji}(a) = \delta_{ii'}\delta_{jj'}$ if $D = D'$!

Problem 32 — Characters

The character of a representation $D : G \to \text{GL}(V)$ is a map

$$\chi_D : G \to \mathbb{C} \,, \qquad a \mapsto \chi_D(a) = \text{tr} D(a) \,.$$

Show the following propositions:

a) Equivalent representations have the same character.

b) The character of a direct sum of representations is the sum of their characters.

c) $\chi(ab) = \chi(ba)$.

d) $\chi(bab^{-1}) = \chi(b)$, i.e., χ is a "class function" and constant on a conjugacy class.

e) $\chi(e) = \dim V$.

Show that for the characters $\chi^{(\alpha)}$ of irreducible representations $D^{(\alpha)}$:

$$\frac{1}{|G|} \sum_{a \in G} \chi^{(\alpha)}(a^{-1}) \chi^{(\beta)}(a) = \delta_{\alpha\beta}$$

($\alpha = \beta$ means equivalent representations).

Show that the multiplicity n_α in a decomposition of a unitary representation

$$D = n_1 D^{(1)} \oplus \cdots \oplus n_r D^{(r)}$$

into irreducible representations $D^{(\alpha)}$ is given by:

$$n_\alpha = \langle \chi^{(\alpha)} | \chi \rangle$$

where $\chi = \chi_D$ and where $\chi = \sum_{a \in G} \chi(a) a \in F_G$ is an element of the Frobenius algebra.