Symmetry Groups in Physics: Problems

Problem 27 — Direct sum and direct product
Let V_1, V_2 be subspaces of a linear space V such that $V = V_1 \oplus V_2$. Let $D_1 : G \to GL(V_1)$ and $D_2 : G \to GL(V_2)$ be two linear representations of a group G in V_1 and in V_2, respectively. Show that $D_1 \oplus D_2$ is a homomorphism!

Let V_1, V_2 be two linear spaces and $D_1 : G \to GL(V_1)$ and $D_2 : G \to GL(V_2)$ be two linear representations of a group G in V_1 and in V_2, respectively. Show that $D_1 \otimes D_2$ is a homomorphism!

Problem 28 — Frobenius Algebra
Consider a finite group G and the group elements $a_1, ..., a_n \in G$ with $n = |G|$ as a basis of an n-dimensional linear space F_G over $\mathbb{C}$! A typical element $x \in F_G$ has the form:

$$x = \sum_{i=1}^{n} x_i a_i$$

with $x_i \in \mathbb{C}$ or

$$x = \sum_{a \in G} x(a)\, a \ ,$$

where $x(a) \in \mathbb{C}$ are the complex coefficients of the expansion of x in the basis. Addition and multiplication with scalars are defined component-by-component.

a) Argue that F_G is in fact a linear space!

b) For elements $x, y \in F_G$ we can define a product via

$$x \cdot y = \left(\sum_a x(a)\, a \right) \left(\sum_b y(b)\, b \right) = \sum_{a,b} x(a) y(b) ab \, .$$

Show that this product is (i) bilinear, (ii) associative and that (iii) there is a neutral element e such that $ex = xe = x \ \forall x \in F_G$! The linear space with this product is a unitary, associative algebra over $\mathbb{C}$!

c) Show that

$$(xy)(a) = \sum_b x(b)\ y(b^{-1}a)$$

for the components of xy, x, y!

d) For elements $x, y \in F_G$ we can define an inner product via

$$\langle x|y \rangle = \frac{1}{|G|} \sum_a x(a)^*\, y(a) \ .$$

Show that the postulates for an inner product are satisfied!

e) Show that $x \in F_G$ can be written as

$$x = |G| \sum_a \langle a|x\rangle\, a\ !$$

f) We define the regular (left) representation of G on F_G,

$$R : G \to \mathsf{GL}(F_G)\ , \quad a \mapsto R(a)$$

by

$$R(a) : F_G \to F_G\ , \quad x \mapsto R(a)x = ax\ .$$

For $a \in G$, $R(a)$ is the left translation!

Show that $R(a)$ is linear!

Show that R is a homomorphism!

Show that

$$R(a)x = \sum_b x(a^{-1}b)b\ !$$

Show that R is unitary, i.e.

$$\langle R(a)x|R(a)y\rangle = \langle x|y\rangle\ !$$

g) Since the elements of G are a basis of F_G, the regular representation can be used to obtain a matrix representation:

$$G \to \mathsf{GL}(F_G) \to \mathsf{GL}(|G|, \mathbb{C})\ ,\ a \mapsto R(a) \mapsto \underline{R}(a)\ .$$

We choose an orthonormal basis as

$$\left\{ \sqrt{|G|}\, a \,\middle|\, a \in G \right\}\ .$$

Show that the elements of the matrix $\underline{R}(a)$ are given by

$$R_{ij}(a) = |G|\, \langle a_i|R(a)a_j\rangle$$

and that $R_{ij}(a) = 1$ if and only if $a_i = aa_j$ and $R_{ij}(a) = 0$ else. This implies

$$aa_j = \sum_i R_{ij}(a)\, a_i\ .$$