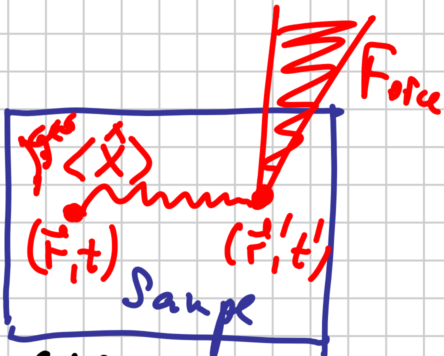
5. Response Function – Vertex

Main experimental scheme:

$\hat{X}$ – "measuring" operator



(ARPES, STM, Optics)

General Perturbation:

$$\hat{H} = \hat{H}_0 + \hat{H}'_t$$

with:

$$\hat{H}'_t = \int d^3r\, F'_i(\vec{r},t) \cdot \hat{X}'_i(\vec{r})$$

example: $F'_i(\vec{r},t) = \varphi(\vec{r},t)$ – electric voltage

$\hat{x}' = \hat{\rho}$ – electronic charge

Effect of perturbation:

H_0: $F'_i = 0 \Rightarrow \langle \hat{X}_i \rangle = 0$

then $F'_i \neq 0 \Rightarrow X(\vec{r},t) = \langle \hat{X}(\vec{r},t) \rangle \neq 0$

In general: functional $X'_i[F'_i]$ – complicated

For small force $\|F'_i\| \ll F_{av} \Rightarrow$

Response is Linear:

$$X_i(\vec{r},t) = \int d^3r' \int dt'\, \chi_{ij}(\vec{r}t,\vec{r}'t') \cdot F'_j(\vec{r}',t') + O(F'^2)$$

$\chi_{ij}(\vec{r}t,\vec{r}'t')$ – Response Function – describe the System (not depends on F')

– can be compare with Theory!

General Properties of χ_{ij}

1) Causality:

Generalized Forces $F_j'(\vec{r}',t')$ can Not cause any effects prior to their action:

$$\chi_{ij}(\vec{r}t, \vec{r}'t') = 0 \quad \text{for } t < t'$$

$\Rightarrow$ Response is Retarded

2) Time-invariant

If H_0 is Not time dependent, then

$$\chi_{ij}(\vec{r}t, \vec{r}'t') = \chi_{ij}(\vec{r}, \vec{r}', \underline{\underline{t-t'}})$$

One can do Fourier transform to frequency space

$$X_i(\vec{r}, \omega) = \int d^3r \, \chi_{ij}(\vec{r}, \vec{r}', \omega) \, F_j'(\omega)$$

3) Space-invariant:

If System H_0 is translationally invariant, then: $\chi_{ij}(\vec{r}, \vec{r}', \omega) = \chi_{ij}(\vec{r} - \vec{r}', \omega) \Rightarrow$

Spatial Fourier transform, leads to relations

$$X_i(\vec{q}, \omega) = \chi_{ij}(\vec{q}, \omega) \, F_j'(\vec{q}, \omega)$$

$\chi_{ij}(\vec{q}, \omega)$ – gives the full dispersion relation of excitations in the system

Theory of Linear Response

Assume: $\hat{X}$ is a single-particle operator:

$$\hat{X} = \sum_{\alpha,\beta} \hat{c}^{+}_{\alpha} X_{\alpha\beta} \hat{c}_{\beta}$$

The action of the system is given by:

$$S[c^{*},c,F'] = \underbrace{S_0[c^{*},c]}_{\text{unperturb action}} + \delta S'[c^{*},c,F']$$

The expectation value of $\hat{X}$ is:

$$X(\tau) = \sum_{\alpha,\beta} \langle c^{*}_{\alpha}(\tau) X_{\alpha\beta}\, c_{\beta}(\tau) \rangle_S$$

where:

$$\langle \ldots \rangle_S = Z^{-1} \int D[c^{*},c] \ldots e^{-S[c^{*},c,F']}$$

is functional average over S-action.

Perturbation action:

$$\delta S'[c^{*},c,F'] = \int d\tau\, H'_{F'} = \int_0^{\beta} d\tau\, F'(\tau) \sum_{\alpha,\beta} c^{*}_{\alpha}(\tau) X'_{\alpha\beta}\, c_{\beta}(\tau)$$

We represent $X(\tau)$ as a derivative of free-energy functional. Let's formally couple operator $\hat{X}$ to a "second" force:

$$\delta S[c^{*},c,F] = \int d\tau\, F(\tau) \cdot \hat{X}(\tau) = \int_0^{\beta} d\tau\, F(\tau) \sum_{\alpha\beta} c^{*}_{\alpha}(\tau) X_{\alpha\beta}\, c_{\beta}(\tau)$$

with:

$$S[c^{*},c,F,F'] = S_0[c^{*},c] + \delta S[c^{*},c,F] + \delta S'[c^{*},c,F']$$

Then, we have:

$$X(\tau) = \frac{\delta \ln Z[F,F']}{\delta F(\tau)}\Big|_{F=0}$$

where partition function:

$$Z[F,F'] = \int D[c^*,c]\, e^{-S[c^*,c,F,F']}$$

Check:

$$\underset{F'}{X}(\tau) = \frac{1}{Z[0,0]} \int D[c^*,c] \sum_{\alpha\beta} c^*_\alpha(\tau) X_{\alpha\beta}\, c_\beta(\tau)\, e^{-S[c^*,c,F,F']}\Big|_{F=0} = \sum_\psi \langle c^* X c\rangle$$

We assume, that F and F' are small $\Rightarrow$

Formal 1^{st}-order expansion,

$$\underset{-}{X}[F'] \simeq \underbrace{\underset{-}{X}[0]}_{=0} + \int d\tau' \frac{\delta X[F']}{\delta F'(\tau)}\Big|_{F'=0} F'(\tau')$$

we can combine:

$$\underset{-}{X}(\tau) \simeq \int d\tau' \Bigg(\underbrace{\frac{\delta^2 \ln Z[F,F']}{\delta F(\tau)\, \delta F'(\tau')}\Big|_{\substack{F=0\\F'=0}}}_{=\chi} \Bigg) F'(\tau')$$

Then the response function:

$$\boxed{\chi(\tau,\tau') = \pm \frac{\delta^2 \ln Z[F,F']}{\delta F(\tau)\, \delta F'(\tau')}\Big|_{\substack{F=0\\F'=0}}}$$

$\ominus$ some-times: $H = -\frac{Kx^2}{2}$

Let us carry out 2-n variation:

$$\chi(\tau,\tau') = \frac{\delta}{\delta F(\tau)}\left[\frac{1}{Z[F,0]}\,\frac{\delta Z[F,F']}{\delta F'(\tau')}\Big|_{F'=0}\right]_{F=0} =$$

$$= Z^{-1}_{[0,0]}\,\frac{\delta^2 Z[F,F']}{\delta F(\tau)\,\delta F'(\tau')}\Big|_{\substack{F'=0\\F=0}} -$$

$$-\underbrace{\left(Z^{-1}_{[0,0]}\,\frac{\delta Z[0,F']}{\delta F'(\tau')}\Big|_{F'=0}\right)}_{=\langle X(\tau')\rangle}\underbrace{\left(Z^{-1}_{[0,0]}\,\frac{\delta Z[F,0]}{\delta F(\tau)}\Big|_{F=0}\right)}_{=\langle X(\tau)\rangle}$$

The 2-nd term is: $\langle X(\tau)\rangle\cdot\langle X(\tau')\rangle = 0$

In this case response function is:

$$\chi(\tau,\tau') = Z^{-1}\,\frac{\delta^2 Z[F,F']}{\delta F(\tau)\,\delta F'(\tau')}\Big|_{\substack{F=0\\F'=0}}$$

Performing the 2nd variation, we get:

$$\chi(\tau,\tau') = Z^{-1}\int \mathcal{D}[c^*,c]\left(\sum_{\alpha\beta} c^*_\alpha(\tau)X_{\alpha\beta}c_\beta(\tau)\right)\left(\sum_{\gamma\delta} c^*_\gamma(\tau')X_{\gamma\delta}c_\delta(\tau')\right)e^{-S'}\Big|_{\substack{F=0\\F'=0}} =$$

$$= \left\langle\left(\sum_{\alpha\beta} c^*_\alpha(\tau)X_{\alpha\beta}c_\beta(\tau)\right)\left(\sum_{\gamma\delta} c^*_\gamma(\tau')X_{\gamma\delta}c_\delta(\tau')\right)\right\rangle_S$$

$\Rightarrow$ 4-point correlation function!

In the case, where $\langle\hat{X}\rangle \neq 0$

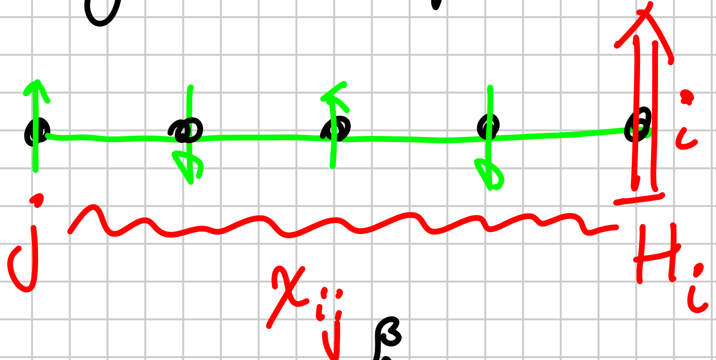
We can write a general irreducible correlation

$$\underbrace{\chi(\tau,\tau')}_{\text{dissipation}} = \Big\langle \underbrace{\left(\hat{X}(\tau)-\langle X(\tau)\rangle\right)\left(\hat{X}(\tau')-\langle X(\tau')\rangle\right)}_{\text{quantum thermal fluctuation}} \Big\rangle$$

This is connection between two different physical properties $\Rightarrow$ fluctuation-dissipation theorem.

Example Magnetic response: (Ising)

$M_j = \chi_{ji} H_i$



(Heisenberg)
$H_i \to H_i^\alpha$
$M_i \to M_i^\alpha$
$\alpha = (x, y, z)$

$$Z[H] = \int \mathcal{D}[c^*, c]\, e^{-S + \int_0^\beta d\tau \sum_i M_i(\tau) H_i(\tau)}$$

where $\quad M_i(\tau) = \sum_{\sigma\sigma'} c^*_{i\sigma}(\tau)\, \sigma^z_{\sigma\sigma'}\, c_{i\sigma'}(\tau)$

Correlations Function:

$$\langle M_i(\tau)\rangle = \left.\frac{\delta \ln Z[H]}{\delta H_i(\tau)}\right|_{H=0}$$

$$\chi_{ij}(\tau-\tau') \equiv \langle M_i(\tau)\cdot M_j(\tau')\rangle = \left.\frac{\delta^2 \ln Z[H]}{\delta H_i(\tau)\,\delta H_j(\tau')}\right|_{H=0}$$

$$\chi_N = \langle M(1) M(2) \cdots M(N)\rangle = \left.\frac{\delta^N \ln Z[H]}{\delta H(1)\,\delta H(2)\cdots \delta H(N)}\right|_{H=0}$$

- N-order irreducible correlation function

5.2 Analitic Structure of Correlation function

2-point correlation function:

$$C_{X_1X_2}(\tau_1-\tau_2) = \frac{1}{Z}\int \mathcal{D}[\bar{c},c]\, X_1(\tau_1)\cdot X_2(\tau_2)\, e^{-S} =$$

$$= \langle T_\tau \hat{X}_1(\tau_1)\cdot \hat{X}_2(\tau_2)\rangle = \begin{cases} \langle \hat{X}_1(\tau_1)\cdot\hat{X}_2(\tau_2)\rangle, & \text{if } \tau_1 \geq \tau_2 \\ \zeta_X \langle \hat{X}_2(\tau_2)\cdot \hat{X}_1(\tau_1)\rangle, & \text{if } \tau_1<\tau_2 \end{cases}$$

$$\zeta_X = \begin{cases} +1, & X=\text{Boson} \\ -1, & X=\text{Fermion} \end{cases}$$

Lehmann representation of correlation function

(Harry Lehmann [1924 – 1998] II-Inst. Theor. Physik)

We use exact eigenbasis of $\hat{H}$:

$$\hat{H}|n\rangle = E_n|n\rangle$$

and

$$\sum_n |n\rangle\langle n| = \hat{\mathbb{1}}$$

If: Grand Canonical:

$$\begin{cases} \hat{H}\to\hat{H}-\mu\hat{N} \\ E_n\to E_n-\mu N_n \end{cases}$$

As example, we proof this for single-particle GF = correlation function for 1-Fermion:

General def. of single-particle correlation function:

$$G_{ij}(\tau-\tau') = -\langle T_\tau \hat{c}_i(\tau)\, \hat{c}_j^{+}(\tau')\rangle_S$$

$$\tau'=0, \quad \tau>0$$

For $\tau > 0$:

$$G_{ij}(\tau) = -\langle \hat{c}_i(\tau)\, \hat{c}_j^{+}(0) \rangle =$$

$$= -Z^{-1}\, \mathrm{Tr}\left(e^{-\beta \hat{H}}\, e^{\tau \hat{H}}\, \hat{c}_i\, e^{-\tau \hat{H}}\, \hat{c}_j^{+} \right) =$$

$$= -Z^{-1} \sum_n \langle n | e^{-\beta \hat{H}}\, e^{\tau \hat{H}}\, \hat{c}_i\, e^{-\tau \hat{H}}\, \hat{\mathbb{1}}\, \hat{c}_j^{+} | n \rangle =$$

$$= -Z^{-1} \sum_{n,m} e^{-\beta E_n} \langle n | e^{\tau \hat{H}}\, \hat{c}_i\, e^{-\tau \hat{H}} | m \rangle \langle m | \hat{c}_j^{+} | n \rangle =$$

$$= -Z^{-1} \sum_{n,m} e^{-\beta E_n}\, e^{\tau (E_n - E_m)} \langle n | \hat{c}_i | m \rangle \langle m | \hat{c}_j^{+} | n \rangle$$

Matsubara-space $\Rightarrow$ Fourier-Transform:

$$G_{ij}(\omega_n) = \int_0^\beta d\tau\, e^{i\omega_n \tau}\, G_{ij}(\tau) =$$

$$= -Z^{-1} \sum_{n,m} \langle n | \hat{c}_i | m \rangle \langle m | \hat{c}_j^{+} | n \rangle\, e^{-\beta E_n} \int_0^\beta d\tau\, e^{\tau (i\omega_n + E_n - E_m)} =$$

$$= -Z^{-1} \sum_{n,m} \langle n | \hat{c}_i | m \rangle \langle m | \hat{c}_j^{+} | n \rangle\, e^{-\beta E_n}\, \frac{e^{i\beta\omega_n}\, e^{\beta(E_n - E_m)} - 1}{i\omega_n + E_n - E_m}$$

note,

$$e^{i\beta\omega_n} = \begin{cases} +1, & \text{Boson}, \ \omega_n = \frac{2n\pi}{\beta} \\ -1, & \text{Fermion}, \ \omega_n = \frac{(2n+1)\pi}{\beta} \end{cases}$$

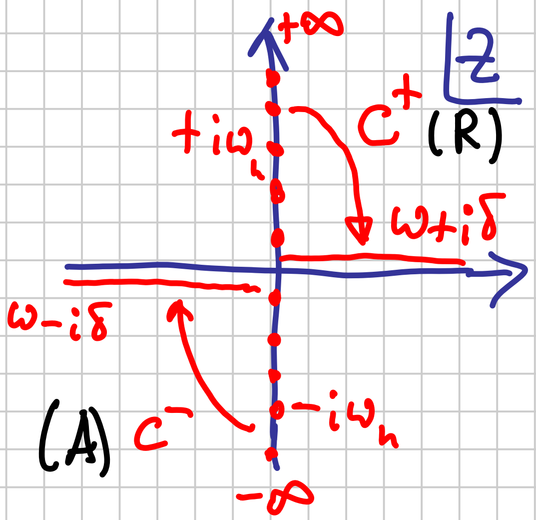
Then:

$$G_{ij}(\omega_n) = Z^{-1} \sum_{n,m} \frac{\langle n | \hat{c}_i | m \rangle \langle m | \hat{c}_j^{+} | n \rangle}{i\omega_n + E_n - E_m} \left(e^{-\beta E_n} + e^{-\beta E_m} \right)$$

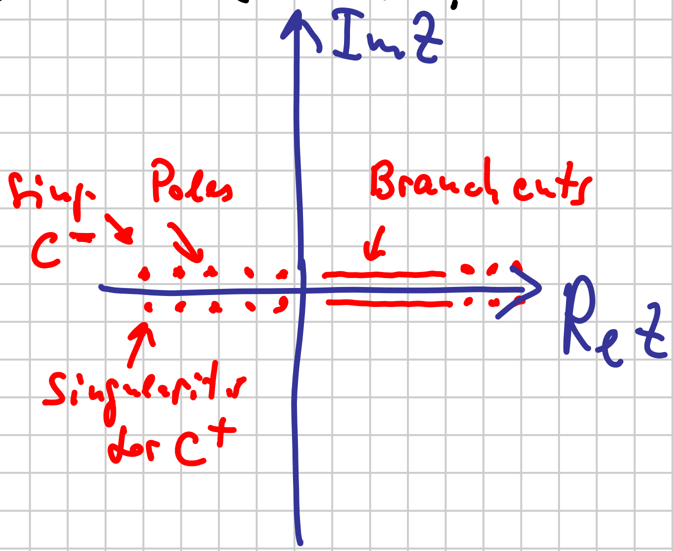
For general correlation function: (Lehmann)

$$\boxed{\begin{aligned} C_{ij}(i\omega_n) &\overset{\text{Def.}}{=} \pm Z^{-1} \sum_{n,m} \frac{X^i_{nm} \cdot X^j_{mn}}{i\omega_n + E_{nm}} \left(e^{-\beta E_n} - \zeta_x e^{-\beta E_m}\right) \\ Z &= \mathrm{Tr}\, e^{-\beta \hat{H}} = \sum_n e^{-\beta E_n} \quad \text{and } E_{nm} = E_n - E_m \end{aligned}}$$

Analitical continuation to real axis:



$\Rightarrow$



Dirac: $$\lim_{\delta \to 0} \frac{1}{x \pm i\delta} = \mp i\pi \delta(x) + \mathcal{P} \frac{1}{x}$$

"Real-time" response function: $\tau \to it$: $C^T(t)$

$$C^T_{ij}(\omega) = \int_{-\infty}^{\infty} dt\, C^T_{ij}(t)\, e^{i\omega t - \delta|t|} = Z^{-1} \sum_{n,m} X^i_{nm} X^j_{mn} \left[\frac{e^{-\beta E_n}}{\omega + E_{nm} + i\delta} - \zeta_x \frac{e^{-\beta E_m}}{\omega + E_{nm} - i\delta}\right]$$

General expression:

$$C^{\left\{\substack{T \\ + \\ -}\right\}}_{ij} = Z^{-1} \sum_{n,m} X^i_{nm} \cdot X^j_{mn} \left[\frac{e^{-\beta E_n}}{\omega + E_{nm} \left\{\substack{+ \\ + \\ -}\right\} i\delta} - \zeta_x \frac{e^{-\beta E_m}}{\omega + E_{nm} \left\{\substack{- \\ + \\ -}\right\} i\delta}\right]$$

Analytical Properties of $C_{ij}(\tau)$

$$\mathrm{Re}\, C^T(\omega) = \mathrm{Re}\, C^+(\omega) = \mathrm{Re}\, C^-(\omega)$$

$$\mathrm{Im}\, C^T(\omega) = \pm\, \mathrm{Im}\, C^{\pm}(\omega) \cdot \begin{cases} \coth \frac{\beta\omega}{2}, & \text{Bosons} \\ \tanh \frac{\beta\omega}{2}, & \text{Fermions} \end{cases}$$

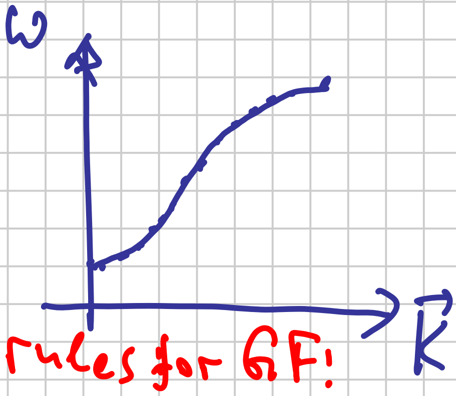
in τ-space! $\quad C(\tau) = \zeta_x\, C(\tau+\beta)$

Spectral Function

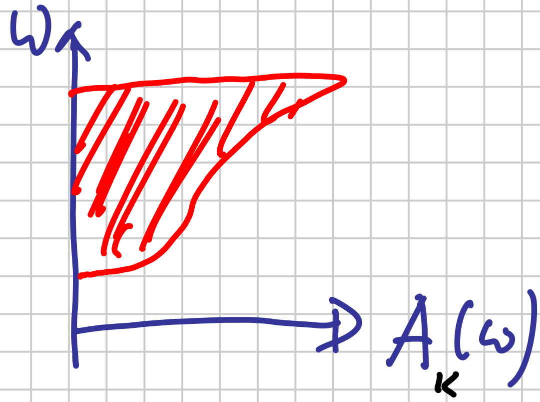
$$A(\omega) = -2\, \mathrm{Im}\, C^+(\omega)$$

using Lehmann-spectral representation,

$$A_{ij}(\omega) = \frac{2\pi}{Z} \sum_{n,m} X^i_{nm} X^j_{mn} \left[e^{-\beta E_n} - \zeta_x e^{-\beta E_m} \right] \delta(\omega + E_{nm})$$



$\Rightarrow$



Sum-rules for GF!

$$\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} A_k(\omega) = Z^{-1} \sum_{n,m} \langle n|\hat{c}_k|m\rangle\langle m|\hat{c}_k^{\dagger}|n\rangle \left[e^{-\beta E_n} - \zeta e^{-\beta E_m} \right] =$$

$$= Z^{-1} \left\{ \sum_n \langle n|\hat{c}_k \hat{c}_k^{\dagger}|n\rangle e^{-\beta E_n} - \zeta \sum_{m \to n} \langle m| c_k^{\dagger} c_k |m\rangle e^{-\beta E_m} \right\} =$$

$$= Z^{-1} \sum_n e^{-\beta E_n} \langle n| \underbrace{\hat{c}_k \hat{c}_k^{\dagger} - \zeta \hat{c}_k^{\dagger} \hat{c}_k}_{[\hat{c}_k, \hat{c}_k^{\dagger}]_\zeta = 1} |n\rangle = Z^{-1} \sum_n e^{-\beta E_n} = 1$$

Fermi - or - Bose - distribution: $n_\zeta(\omega) = \frac{1}{e^{\beta\omega} - \zeta}$

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} n_\zeta^{F/B}(\omega) A_k(\omega) = Z^{-1} {\sum_{n,m}}' \langle n|\hat{c}_k|m\rangle\langle m|\hat{c}_k^\dagger|n\rangle \left[e^{-\beta E_n} - \zeta e^{-\beta E_m}\right] \int d\omega \frac{\delta(\omega + E_{nm})}{e^{\beta\omega} - \zeta}$$

$$= Z^{-1} {\sum_{n,m}}' \langle n|c_k|m\rangle\langle m|c_k^\dagger|n\rangle e^{-\beta E_n}\left[e^{\beta E_{nm}} - \zeta\right] \frac{1}{e^{\beta E_{nm}} - \zeta}$$

$$= Z^{-1} \sum_m e^{-\beta E_m} \langle m|\hat{c}_k^\dagger \hat{c}_k|m\rangle = \langle \hat{n}_k \rangle$$

Kramer - Kronig relation

$$A(\omega) = i\left[C^+(\omega) - C^-(\omega)\right]$$

then we obtain:

$$C(z) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{A(\omega)}{z-\omega} \overset{\mathrm{Im}\,z>0}{=} -\frac{1}{2\pi i} \int d\omega \frac{C^+}{z-\omega}$$

Consider: $z = \omega^+ \equiv \omega + i0$ $\delta \to 0$

$$C^+(\omega) = -\frac{1}{2\pi i} \int d\omega' \frac{C^+(\omega')}{\omega - \omega' + i\delta} \overset{\text{Dirac}}{=} \frac{1}{\pi i} \int d\omega' C^+(\omega') \mathcal{P} \frac{1}{\omega' - \omega}$$

Then:

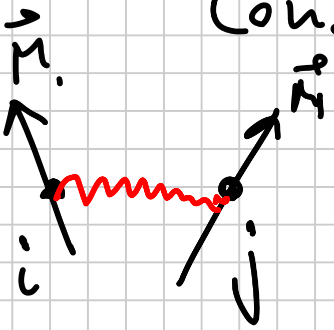
$$\mathrm{Re}\, C^+(\omega) = \frac{1}{\pi} \fint d\omega' \frac{\mathrm{Im}\, C^+(\omega')}{\omega' - \omega}$$

$$\mathrm{Im}\, C^+(\omega) = -\frac{1}{\pi} \fint d\omega' \frac{\mathrm{Re}\, C^+(\omega')}{\omega' - \omega}$$

KK

5.3 General expression for Response Function

Consider Magnetic Response:



$$\chi_{ij}^{\alpha\beta}(\tau-\tau') = \left\langle \hat M_i^{\alpha}(\tau)\cdot \hat M_j^{\beta}(\tau')\right\rangle_S$$

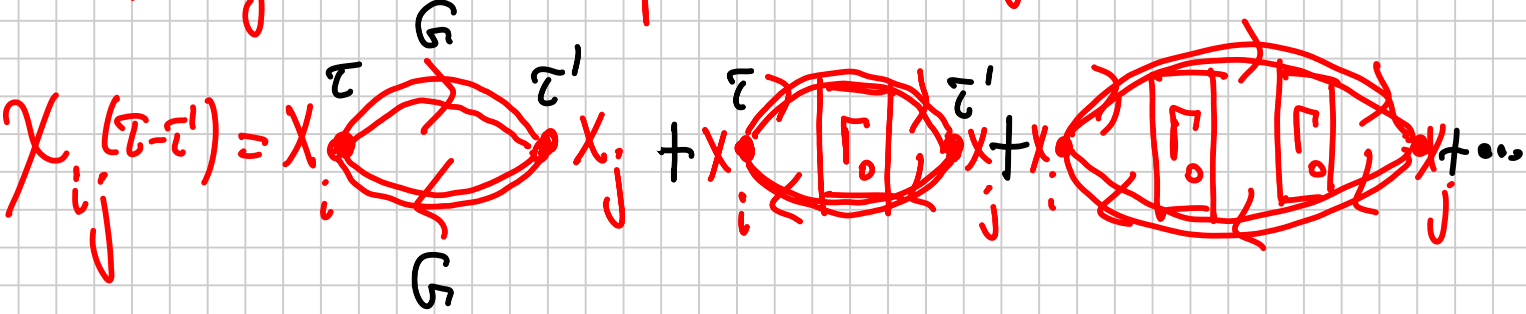
$\alpha = \{x,y,z\}$

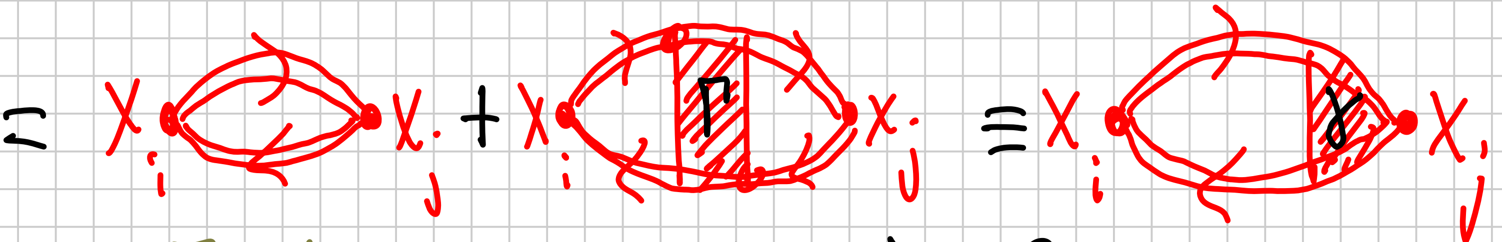
$$\hat M_i^{\alpha} = \sum_{\sigma\sigma'} \hat c^{+}_{i\sigma}\,\sigma^{\alpha}_{\sigma\sigma'}\,\hat c_{i\sigma'}$$

Simple notation: $\hat M_i = \sum_{\dots} \hat c_i^{+}\,\underline{X}_i\,\hat c_i$

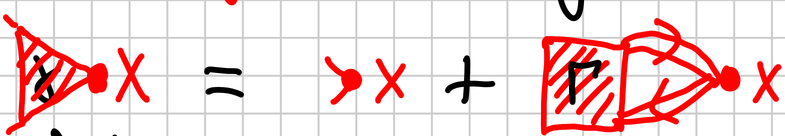
$$\chi_{ij}(\tau-\tau') = \underline{X}_i \left\langle c_i^{*}(\tau)\, c_i(\tau)\, c_j^{*}(\tau')\, c_j(\tau')\right\rangle_S X_j$$

Diagrammatic expansion: $i \neq j$





Γ = 4-point (leg) vertex, γ = 3 point (leg) vertex:



$$\tilde X = \gamma\cdot X = X + \Gamma * GG\cdot X$$

$$\gamma = 1 + \Gamma * GG$$

Connection between χ and G^{II} (carefully)

$$G^{II}_{1234} = \langle c_1 c_2 c^*_3 c^*_4 \rangle \overset{\Gamma}{=}$$

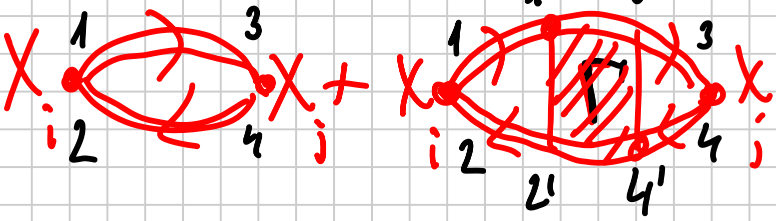
$=$ 

(p h)

(p-p)

$$\chi_{ij}(\tau,\tau') = \sum_{1234} \langle 1|X^i|2\rangle \chi_{1234} \langle 4|X^j|3\rangle$$

$$\chi_{1234} = -\left\{ \langle c^*_1 c_2 c^*_3 c_4 \rangle - \langle c^*_1 c_2 \rangle \langle c^*_3 c_4 \rangle \right\} =$$

$=$ 

$=$

$$\overset{X=1}{=} G_{13} * G_{42} + G_{11'} * G_{2'2} * \Gamma_{1'2'3'4'} * G_{3'3}\, G_{44'} =$$

$$\chi_{12,34} = \chi^0_{12,34} + \chi^0_{12,1'2'} * \Gamma_{1'2',3'4'} * \chi^0_{3'4',34}$$

Super-index: $i = \{1,2\}$ $j = \{3,4\}$

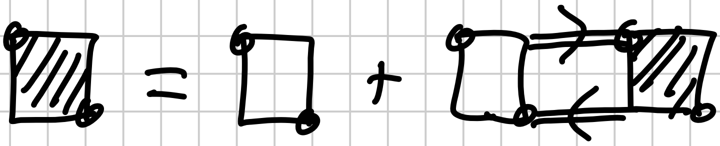
$$\chi_{ij} = \chi^0_{ij} + \chi^0_{ii'} * \Gamma_{i'j'} * \chi^0_{j'j}$$

Matrix Form:

$$\hat{\chi} = \hat{\chi}_0 + \hat{\chi}_0 * \hat{\Gamma} * \hat{\chi}_0$$

where 4-poin Vertex Γ is solution of BSE:

(p-h) channel:



$$\Gamma = \Gamma_0 + \Gamma_0 \chi_0 \Gamma$$

Then we can find equation for $\hat{\chi}$:

$$\hat{\chi} = \hat{\chi}_0 + \hat{\chi}_0 * \hat{\Gamma}_0 * \hat{\chi}$$

or:

$$\hat{\chi}^{-1} = \hat{\chi}_0^{-1} - \hat{\Gamma}_0$$

Examples of Response functions

1) Density response

φ - potential

$$\hat{H}_t' = \int d\tau \int d^3r \, \hat{n}(\vec{r},\tau) \cdot \overset{\varphi}{\hat{V}}(\vec{r},t)$$

$$\hat{n}_i = \sum_\sigma \hat{c}^+_{i\sigma} c_{i\sigma}$$

$$n(\vec{r},\tau) = \int d\tau' \int d^3r' \, \chi(\vec{r},\tau,\vec{r}',\tau') V(\vec{r}',t') \quad X_{\alpha\beta} = 1$$

$$\chi_{ij}(\tau,\tau') = \langle n_i(\tau) \cdot n_j(\tau') \rangle - \langle n_i \rangle \langle n_j \rangle$$

2) Electromagnetic Response

$$\hat{H}_t' = \int d^4x \, \hat{j}_\mu(x) \cdot A^\mu(x)$$

$$x = (\tau, \vec{r}), \quad \hat{j}^\mu = (\hat{n}, \hat{\vec{j}}), \quad A^\mu = (\underset{\varphi}{V}, \vec{A})$$

then the linear response:

$$j_\mu(x) = \int d^4x' \, K_{\mu\nu}(x,x') A^\nu(x')$$

General definition of E-M response:

$$K_{\mu\nu}(x,x') = \left. \frac{\delta^2 \ln Z[A]}{\delta A_\mu(x) \, \delta A_\nu(x')} \right|_{A=0}$$

$$S[c^*,c,A] = \int d^4x \, c^* \left[\partial_\tau - \mu + V + \frac{1}{2m}\left(-i\vec{\nabla} - \vec{A}\right)^2 \right] c + S_{int}[c^*,c]$$

then:

$$K_{\mu\nu}(x,x') = - \underbrace{\frac{\langle \hat{n}(x) \rangle}{m} \delta(x-x') \delta_{\mu\nu}(1-\delta_{\mu 0})}_{\text{diamagnetic}} + \underbrace{\langle j_\mu(x) \, j_\nu(x') \rangle}_{\text{paramagnetic}}$$

diamagnetic

paramagnetic

optical response $\sigma(x)$