4. Green Function and Perturbation Theory

4.1. GF for interacting system.

$$\hat{H} = \hat{H}_0 + \hat{V}$$

if basis $|i\rangle$ choose to diagonalized H_0:

$$\hat{H}_0 = \sum_i \varepsilon_i \hat{c}^\dagger_i \hat{c}_i$$

general 2-particle interactions:

$$\hat{V} = \frac{1}{2} \sum_{ijkl} V_{ijkl}\, \hat{c}^\dagger_i \hat{c}^\dagger_j \hat{c}_l \hat{c}_k$$

Path-integral for partition function:

$$Z = \int\limits_{c(\beta)=\zeta c(0)} D[c^*, c]\, e^{-S}$$

with action: $c_i \equiv c_i(\tau)$ $(\mu = 0)$

$$S = \int_0^\beta d\tau \left[\sum_i c^*_i (\partial_\tau + \varepsilon_i) c_i + \frac{1}{2} \sum_{ijkl} V_{ijkl}\, c^*_i c^*_j c_l c_k \right]$$

Zero-order partition function $(\hat{V} = 0, \Rightarrow Z_0)$

$$Z_0 = \int D[c^*, c]\, e^{-S_0}$$

$$S_0 = \int_0^\beta d\tau \sum_i c^*_i (\partial_\tau + \varepsilon_i) c_i$$

Exact PI-calculation: $Z_0 = \prod_i \left(1 - \zeta e^{-\beta \varepsilon_i}\right)$

Define thermal average with respect to S_0 and S:

$\forall F(\hat{c}^+, \hat{c})$:
$$\langle F \rangle = \frac{1}{Z} \int D[c^*, c]\, F(c^*, c)\, e^{-S}$$
$$\langle F \rangle_0 = \frac{1}{Z_0} \int D[c^*, c]\, F(c^*, c)\, e^{-S_0}$$

Define 1-particle and 2-particle Green Functions:

1-p: $$G^{I}_{ij}(\tau_1, \tau_2) = -\langle T_\tau \hat{c}_i(\tau_1) \hat{c}^+_j(\tau_2) \rangle = -\frac{1}{Z} \int D[c^*, c]\, c_i(\tau_1) c^*_j(\tau_2)\, e^{-S}$$

2-p: $$G^{II}_{ijkl}(\tau_1, \tau_2, \tau_3, \tau_4) \equiv K_{ijkl} = \langle T_\tau\, \hat{c}_i(\tau_1)\, \hat{c}_j(\tau_2)\, \hat{c}^+_k(\tau_3)\, \hat{c}^+_l(\tau_4) \rangle =$$
$$= \frac{1}{Z} \int D[c^*, c]\, c_i(\tau_1) c_j(\tau_2) c^*_k(\tau_3) c^*_l(\tau_4)\, e^{-S}$$

General n-point correlator:
$$G^n(\tau_1 \cdots \tau_n) = \underbrace{(-1)^n}_{=\,?} \langle c_1 \cdots c_n\, c^*_{n+1} \cdots c_{2n} \rangle$$

Main idea: expand around Gaussian PI: S_0

We can rewrite Z as:
$$Z = \int D[c^*, c]\, e^{-\int_0^\beta d\tau \left[\sum_i c^*_i (\partial_\tau + \varepsilon_i - \mu) c_i + V(c^* c^* c c)\right]} =$$
$\Rightarrow e^{S_0} \cdot e^{\int V d\tau}$
$$= Z_0 \left\langle e^{-\int_0^\beta d\tau\, V(c^*, c^*, c, c)} \right\rangle_0$$

$\Rightarrow$ expand exp!

Perturbation expansion in power of V:

$$\frac{Z}{Z_0} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^{\beta} \cdots \int d\tau_1 \cdots d\tau_n \left\langle V(c_1^* c_1^* c_1 c_1) \cdots V(c_n^* c_n^* c_n c_n) \right\rangle_0$$

it is possible to find exact integral:

$$\langle c_1 \cdots c_n \, c_{n+1}^* \cdots c_{2n}^* \rangle_0 \Rightarrow \text{Gaußian P.I.}$$

4.2 Wick's theorem

General form of non-interacting $\hat{H}_0$:

$$\hat{H}_0 = \sum_{\alpha,\beta} t_{\alpha\beta} \hat{c}_\alpha^+ \hat{c}_\beta$$

Partition function:

$$Z_0 = \int D[c_i^* c] \, e^{-\int_0^\beta d\tau \sum_{\alpha\beta} c_\alpha^*(\tau) \underbrace{\left[(\partial_\tau - \mu)\delta_{\alpha\beta} + t_{\alpha\beta}\right]}_{M_{ij}} c_\beta(\tau)}$$

Short notation: $i = \{\alpha, \tau\}$ $\quad \int d\tau \sum_\alpha \longrightarrow \sum_i$

Partition function with the source terms: J^*, J

$J^* c \longrightarrow c^* J$

$$Z_0[J^*, J] = \int D[c^*, c] \, e^{-\sum_{ij} c_i^* M_{ij} c_j + \sum_i (J_i^* c_i + c_i^* J_i)} \overset{\text{Gauß}}{=} [\det M]^{-\xi} \, e^{\sum_{ij} J_i^* M_{ij}^{-1} J_j}$$

Define Generating function:

$$G[J^*,J] = \frac{\int D[c^*,c]\, e^{-\sum_{ij} c_i^* M_{ij} c_j + \sum_i (J_i^* c_i + c_i^* J_i)}}{\int D[c^*,c]\, e^{-\sum_{ij} c_i^* M_{ij} c_j}} = e^{\sum_{ij} J_i^* M_{ij}^{-1} J_j}$$

Non-interacting n-particle Green function:

$$G_0^{(n)}(i_1 \dots i_n, j_1 \dots j_n) \overset{\text{def.}}{=} \left.\frac{\delta^{2n} G(J^*,J)}{\delta J_{i_1}^* \cdots \delta J_{i_n}^* \, \delta J_{j_n} \cdots \delta J_{j_1}}\right|_{\substack{J=0\\ J^*=0}} =$$

$$= \xi^n \frac{\int D[c^*,c]\, c_{i_1} \cdots c_{i_n} \cdot c_{j_1}^* \cdots c_{j_n}^* \, e^{-\sum_{ij} c_i^* M_{ij} c_j}}{\int D[c^*,c]\, e^{-\sum_{ij} c_i^* M_{ij} c_j}} =$$

$$= \left.\frac{\delta^{2n}\left(e^{\sum_{ij} J_i^* M_{ij}^{-1} J_j}\right)}{\delta J_{i_1}^* \cdots \delta J_{i_n}^* \, \delta J_{j_n} \cdots \delta J_{j_1}}\right|_{\substack{J=0\\ J^*=0}} = \xi^n \frac{\delta^n}{\delta J_{i_1}^* \cdots \delta J_{i_n}^*} \left(\sum_{k_n} J_{k_n}^* M_{k_n j_n}^{-1}\right) \cdots \left(\sum_{k_1} J_{k_1}^* M_{k_1 j_1}^{-1}\right) \underbrace{e^{\sum_{ij} J_i^* M_{ij}^{-1} J_j}}_{=1}\Bigg|_{\substack{J=0\\ J^*=0}}$$

$$= \xi^n \sum_P \xi^P M_{i_{P_n} j_n}^{-1} \cdots M_{i_{P_1} j_1}^{-1}$$

$\Rightarrow$ Wick's theorem:

The general Wick-theorem:

$$\frac{\int D[c^*,c]\, c_{i_1}\cdots c_{i_n} c^*_{j_n}\cdots c^*_{j_1}\, e^{-\sum_{ij} c^*_i M_{ij} c_j}}{\int D[c^*,c]\, e^{-\sum_{ij} c^*_i M_{ij} c_j}} = \sum_P \zeta^P \overset{g}{\overbrace{M^{-1}_{i_{P_n} j_n}}} \cdots M^{-1}_{i_{P_1} j_1} \overset{F}{\underset{B\to Pf.}{=}} \det(g_{ij})_{n\times n}$$

The 1-particle 0-Green-function $(G_0 = g)$

$$G_0^{(I)}(\alpha_1\tau_1,\alpha_2\tau_2) = \zeta \frac{\int D[c^*,c]\, c_{\alpha_1}(\tau_1)\, c^*_{\alpha_2}(\tau_2)\, e^{-\int_0^\beta d\tau \sum_{\alpha\beta} c^*_\alpha(\tau)[(\partial_\tau-\mu)\delta_{\alpha\beta}+t_{\alpha\beta}]c_\beta(\tau)}}{\int D[c^*,c]\, e^{-\int_0^\beta d\tau\, c^*_\alpha(\tau)[(\partial_\tau-\mu)\delta_{\alpha\beta}+t_{\alpha\beta}]c_\beta(\tau)}} =$$

$$= \zeta\left(\partial_\tau - \mu + \hat{t}\right)^{-1}_{\alpha_1\tau_1,\alpha_2\tau_2} \equiv g_{\alpha_1\alpha_2}(\tau_1-\tau_2)$$

FT: $(\underbrace{\tau_1-\tau_2}_{\tau}) \to \omega_n$

Lattice: $\underbrace{\alpha_1-\alpha_2}_{\vec{R}} \to \vec{K}$

$t_{\alpha\beta} \to t_K \equiv \varepsilon_K$

$$g_{\vec{K}}(\omega_n) \overset{F}{=} \frac{1}{i\omega_n + \mu - \varepsilon_K}$$

For n-particle non-interacting GF:

$$G_0^{(n)}(\underbrace{\alpha_1\tau_1}_{1}\cdots\underbrace{\alpha_n\tau_n}_{n},\underbrace{\alpha'_n\tau'_n}_{n'}\cdots\underbrace{\alpha'_1\tau'_1}_{1'}) \equiv \zeta^n \langle c_1\cdots c_n c^*_{n'}\cdots c_{1'}\rangle =$$

Wick

$$= \zeta^n \sum_P \zeta^P g_{P_1 1'}\cdots g_{P_n n'} = \zeta^n \det(g_{ij})_{n\times n}$$

Example: (F)

$$G_0^{\mathrm{II}} = \langle c_1 c_2 c^*_{2'} c^*_{1'} \rangle = g_{11'} \cdot g_{22'} - g_{21'} \cdot g_{12'} \equiv$$

$$\equiv \det \begin{vmatrix} g_{11'} & g_{12'} \\ g_{21'} & g_{22'} \end{vmatrix}$$

4.3 Perturbation Theory and CT-QMC

$$\hat{H} = \hat{H}_0 + \hat{V}$$

$$\hat{H}_0 = \sum_{\alpha\beta} t_{\alpha\beta}\, \hat{c}^+_\alpha \hat{c}_\beta$$

$$\hat{V} = \frac{1}{2} \sum_{\alpha\beta\gamma\delta} U_{\alpha\beta\gamma\delta}\, \hat{c}^+_\alpha \hat{c}^+_\beta \hat{c}_\delta \hat{c}_\gamma$$

$$\frac{Z}{Z_0} = \left\langle e^{-\int_0^\beta d\tau \frac{1}{2} \sum_{\alpha\beta\gamma\delta} U_{\alpha\beta\gamma\delta}\, c^*_\alpha(\tau)\, c^*_\beta(\tau)\, c_\delta(\tau)\, c_\gamma(\tau)} \right\rangle_0 =$$

wick

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\, 2^n} \sum_{\{\alpha\beta\gamma\delta\}}' \int_0^\beta \cdots \int d\tau_1 \ldots d\tau_n\, U_{\alpha_1\beta_1\gamma_1\delta_1} \cdots U_{\alpha_n\beta_n\gamma_n\delta_n} \langle c^*_1 c^*_1 c_1 c_1 \cdots c^*_n c^*_n c_n c_n \rangle_0 =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\, 2^n} \sum_{\{\alpha\beta\gamma\delta\}}' U_{\alpha_1\beta_1\gamma_1\delta_1} \cdots U_{\alpha_n\beta_n\gamma_n\delta_n} \int_0^\beta \cdots \int d\tau_1 \ldots d\tau_n\, \det |g_0(\tau_i - \tau_j)|_{2n \times 2n}$$

$\Rightarrow$ "exact" perturbation theory

Numerical: CT-QMC $\equiv$ Continuous-time Quantum-Monte-Carlo

MC: $$\sum_n \sum_{\{\alpha\beta\gamma\delta\}} \int_0^\beta \cdots \int d\tau_1 \cdots d\tau_n \equiv \sum_K$$

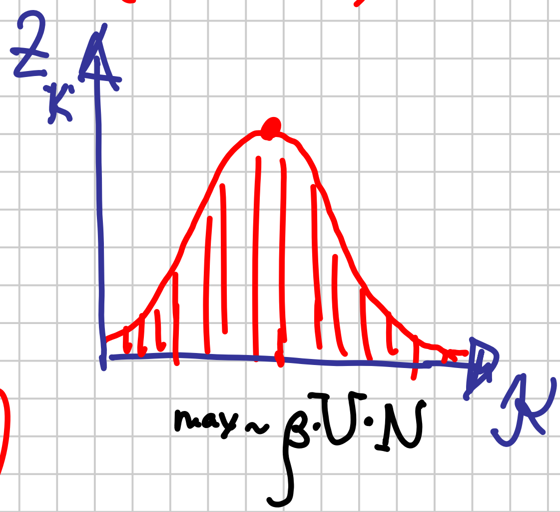
Partition function:
$$Z = Z_0 \sum_n \frac{(\cdot\cdot)}{(\cdot\cdot)} \sum_{\{\alpha\beta\gamma\delta\}} V \cdots V \int_0^\beta \cdots \int d\tau_1 \cdots d\tau_n \det(g)_{2n\times 2n} \equiv \sum_K Z_K$$

Analogy: for 1-particle Green-function: (CT-QMC)

$$G = \sum_K G_K \cdot Z_K$$

MC-important Sampling

Probability weight $(\det g)$



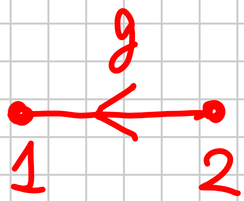
For - Bosons $\Rightarrow$ numerically exact

For - Fermions $\Rightarrow$ the only problem is "fermion sign"

Feynman Diagrams: $|1\rangle = |\alpha, \tau\rangle$

Graphs:

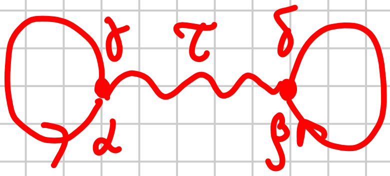
$$G(1,2) = \}\langle c_1 c_2^* \rangle_{S_0} \equiv$$



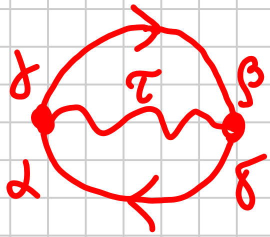
$$V_{1234} = \langle c_1^* c_2^* | \hat{V} | c_3 c_4 \rangle \equiv$$

1-st order perturbation!

$\langle c^{*}_{\alpha}(\tau)\, c^{*}_{\beta}(\tau)\, c_{\delta}(\tau)\, c_{\gamma}(\tau)\rangle_0$



$$= -\frac{1}{2}\sum_{\alpha\beta\gamma\delta} V_{\alpha\beta\gamma\delta}\int_0^\beta d\tau\, \underset{\tau-\tau}{G_{\alpha\gamma}(0)}\cdot G_{\beta\delta}(0)$$



$$= -\frac{1}{2}\zeta\sum_{\alpha\beta\gamma\delta} V_{\alpha\beta\gamma\delta}\int_0^\beta d\tau\, G_{\alpha\delta}(0)\, G_{\beta\gamma}(0)$$

$\langle c^{*}_{\alpha}(\tau)\, c^{*}_{\beta}(\tau)\, c_{\delta}(\tau)\, c_{\gamma}(\tau)\rangle_0$

$G(0) \leftarrow$ propagator of equal time

For diagonal Green-function: $G_{\alpha\beta}(0) = G_\alpha(0)\,\delta_{\alpha\beta} = n_\alpha\,\delta_{\alpha\beta}$

$$\frac{Z}{Z_0} = \overset{\text{0-order}}{1} - \frac{\beta}{2}\sum_{\alpha\beta}\left[\langle\alpha\beta|V|\alpha\beta\rangle + \zeta\langle\alpha\beta|V|\beta\alpha\rangle\right] n_\alpha n_\beta$$

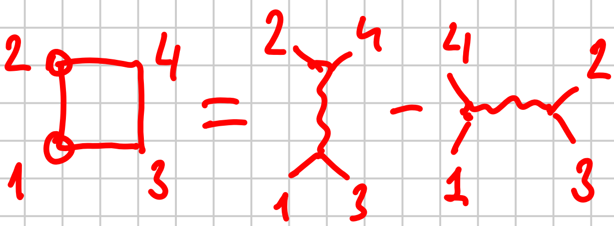
Grand-canonical potential: $\Omega = -\frac{1}{\beta}\ln Z$:

$$\Omega = \Omega_0 + \frac{1}{2}\sum_{\alpha\beta}\left[\langle\alpha\beta|V|\alpha\beta\rangle + \zeta\langle\alpha\beta|V|\beta\alpha\rangle\right] n_\alpha n_\beta$$

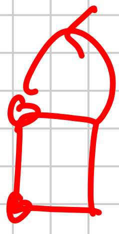
– Hartree-Fock Apr.

For fermions: "loops" gives a factor $\zeta = -1$ (F)

We have to be carefully with "loops" $= (-1)$ high for fermions!



HF:



Higher-order diagrams.

Factors: 1) n-th order $= \dfrac{(-1)^n}{2^n n!}$

2) Wick "permutations" $= (2n)!$

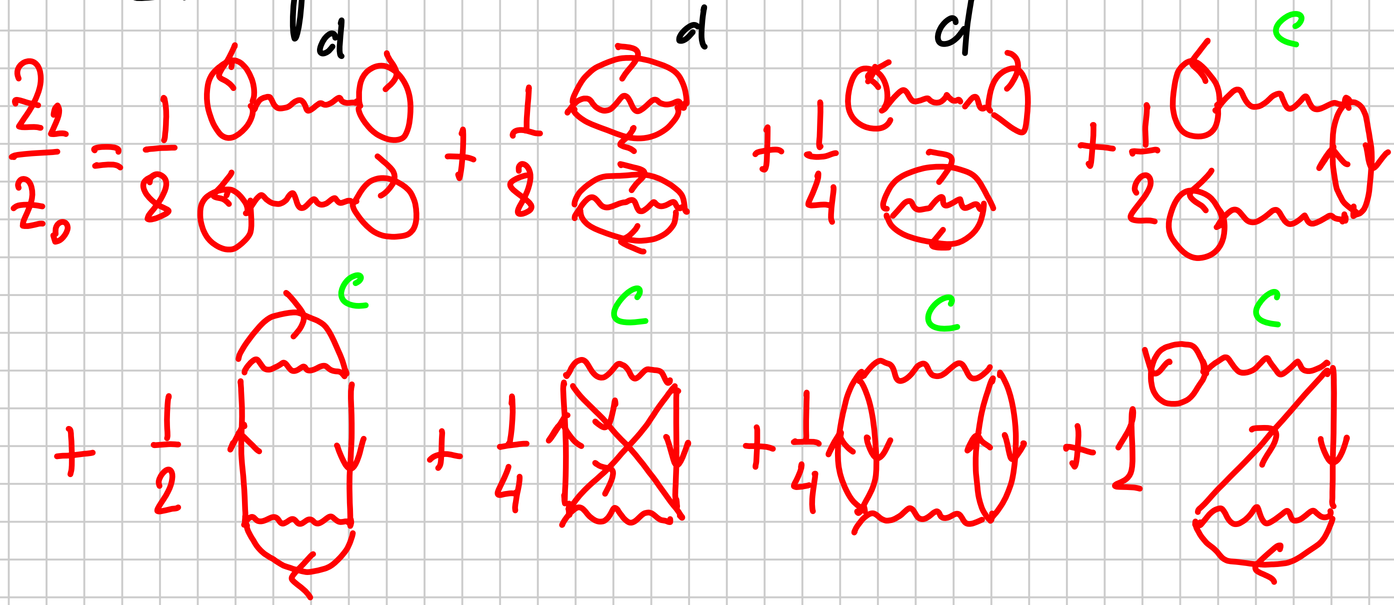
n-th order: $\left.\begin{matrix} 2n \to c \\ 2n \to c^* \end{matrix}\right\}$

Factors $\dfrac{(-1)^n}{S}$

S = symmetry factors of "unlabeled" diagram

Sumrule: $$\sum_{n\text{-order}} \frac{1}{S} = \frac{(2n)!}{2^n n!} = (2n-1)!! \equiv (2n-1)(2n-3)\cdots 5\cdot 3\cdot 1$$

Example: 2-order



$$\sum_{n=2} \frac{1}{S} = 3$$

c = connected

d = disconnected

Diagrams for $\ln Z$: only connected!

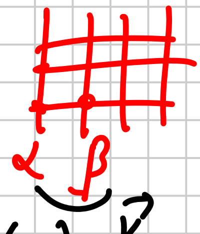
$$\Omega = -\frac{1}{\beta} \ln Z$$

$$G = \frac{\delta \ln Z[J^* J]}{\delta [J^* J]}$$

Dyson Equation

1-particle Green's Function for simple lattice!

$$G_{\alpha\beta}(\tau) \xrightarrow{FT} G(\vec{k},\omega)$$



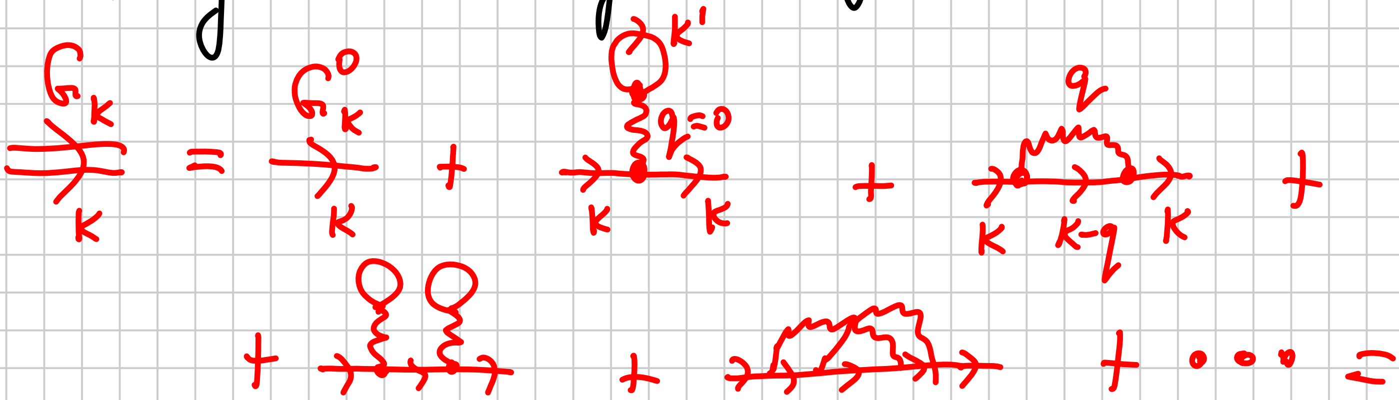
4-notation: $k \equiv (\vec{k},\omega)$ $\qquad V_{\alpha\beta\gamma\delta} \Rightarrow V(q)$ $\vec{k}$

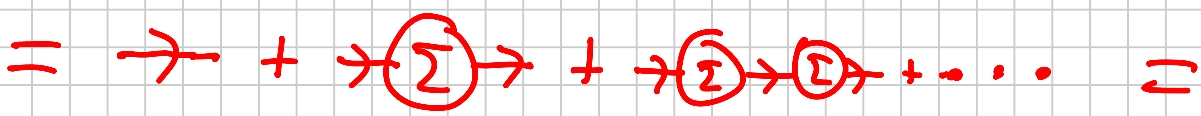
bare: $G_0 \equiv \rightarrow = \{\langle cc^*\rangle_{S_0}$

exact: $G \equiv \Rightarrow = \{\langle cc^*\rangle_{S}$

relation?

Feynman diagrams for G:



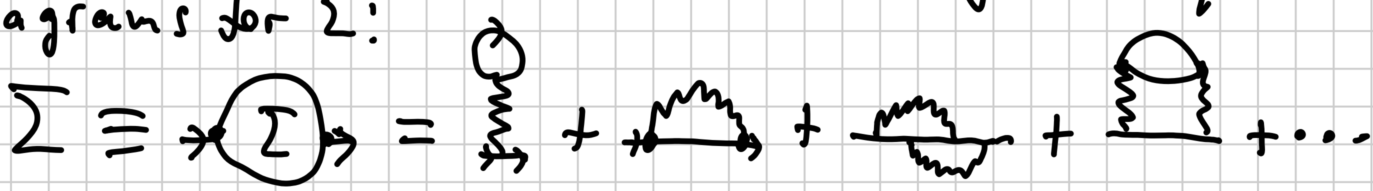


$$\hat{G} = \hat{G}_0 + \hat{G}_0 \otimes \hat{\Sigma} \otimes \hat{G} \qquad \text{Dyson Eq.}$$

$$G(\vec{k},\omega) = G_0(\vec{k},\omega) + G_0(\vec{k},\omega)\cdot\Sigma(\vec{k},\omega)\cdot G(\vec{k},\omega)$$

$$G^{-1} = G_0^{-1} - \Sigma \qquad \text{Dyson Eq.}$$

Diagrams for Σ:



Quasi-particle spectrum (Lattice Fermion)

one-band model

$$\varepsilon_k = \sum_{\langle ij \rangle} t_{ij} e^{i\vec{k}\vec{R}_{ij}}$$

$$G_0(\vec{k}, i\omega) = \frac{1}{i\omega - \varepsilon_{\vec{k}}}$$

$$G(\vec{k}, i\omega) = \frac{1}{i\omega - \varepsilon_{\vec{k}} - \Sigma(\vec{k}, \omega)}$$

Dyson Eq.

Quasiparticle spectrum: $i\omega \to \omega + i\delta$, $\delta \to 0$

$$\Sigma(\vec{k}, \omega) = \underset{(\mathrm{Re})}{\Sigma'(\vec{k}, \omega)} + \underset{(\mathrm{Im})}{\Sigma''(\vec{k}, \omega)}$$

new spectrum:

$$\omega - \varepsilon_k - \Sigma'(\vec{k}, \omega) = 0 \Rightarrow \varepsilon^*_k = \varepsilon_k + \Sigma'(\vec{k}, \varepsilon^*_k)$$

Then Green's Function

$$G(\vec{k}, \omega) = \frac{1}{(\omega - \varepsilon^*_k) - \left.\frac{\partial \Sigma'}{\partial \omega}\right|_{\varepsilon^*_k}(\omega - \varepsilon^*_k) - i\Sigma''} + G_{rest} \equiv$$

$$\equiv \frac{Z_k}{\omega - \varepsilon^*_k - i\gamma_k} + G_{inc.}$$

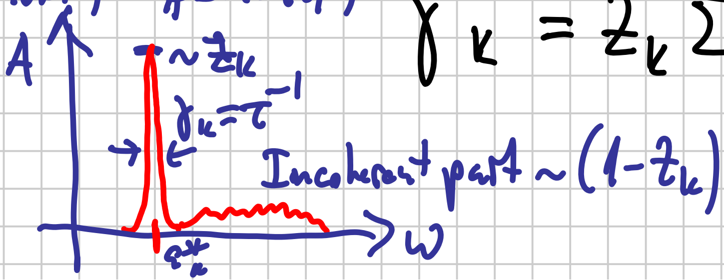
where:

$$Z_k = \left(1 - \frac{\partial \Sigma'}{\partial \omega}\right)^{-1}_{\omega = \varepsilon^*_k} \quad \text{– renormalization factor}$$

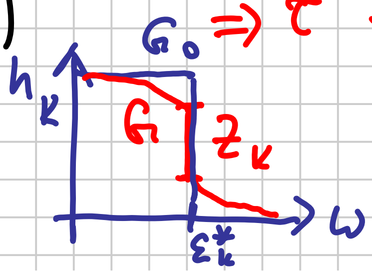
$$\gamma_k = Z_k \Sigma''(\vec{k}, \varepsilon^*_k) \Rightarrow \tau^{-1} \text{ – lifetime}$$

Spectral Function

$$A(\vec{k}, \omega) = -\frac{1}{\pi} \mathrm{Im}\, G(\vec{k}, \omega)$$



$\Longrightarrow$



The Linked Cluster Theorem

$$\ln Z = \lim_{n\to 0} \frac{d}{dn}\left(e^{n \ln Z}\right) = \lim_{n\to 0} \frac{d}{dn} Z^n$$

we first evaluate Z^n for integer n, then

$\ln Z \sim$ coef. of the Graphs proportional to n.

$$\left(Z^n = c\cdot n \Rightarrow \ln Z = c \right)$$

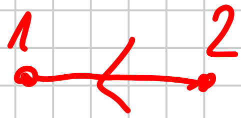
Since in Path Integral:

$$\frac{Z}{Z_0} = \frac{1}{Z_0}\int D[c^*_\alpha, c_\alpha]\, e^{-\int_0^\beta d\tau \left\{ \sum_\alpha c^*_\alpha(\tau)\left[\partial_\tau - \mu + \varepsilon_\alpha\right] c_\alpha(\tau) + V(c^*_\alpha c^*_\alpha c_\alpha c_\alpha)\right\}}$$

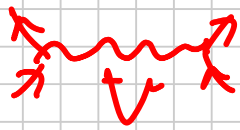
Then we can write Z^n as a Path-Int. over n- sets of fields: $\{c^{*\sigma}_\alpha, c^\sigma_\alpha\}$, where $\sigma = 1, \dots n$

$$\left(\frac{Z}{Z_0}\right)^n = \frac{1}{Z_0^n}\int \prod_{\sigma=1}^n D[c^{*\sigma}_\alpha, c^\sigma_\alpha]\, e^{-\int_0^\beta d\tau \sum_{\sigma=1}^n \left\{ \sum_\alpha c^{*\sigma}_\alpha(\tau)\left[\partial_\tau - \mu + \varepsilon_\alpha\right] c^\sigma_\alpha(\tau) + V(c^{*\sigma}_\alpha c^{*\sigma}_\alpha c^\sigma_\alpha c^\sigma_\alpha)\right\}}$$

$\Rightarrow$ Feynman Diagram for Z^n:



$\sim G^\sigma_\alpha(\tau_1 - \tau_2)$



$\sim V$ - same for σ (save)

Each "connected" part carry a single index σ

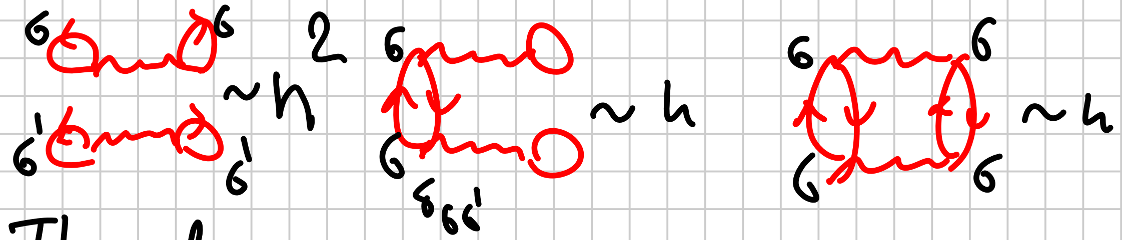
$\Rightarrow$ then $\sum_{\sigma=1}^n 1 = n$

Graph with m connected part $\sim n^m$

Thus Graphs $\sim n$ should have one connected part

$\Rightarrow$ that is connected diagrams

$\Sigma \Rightarrow$



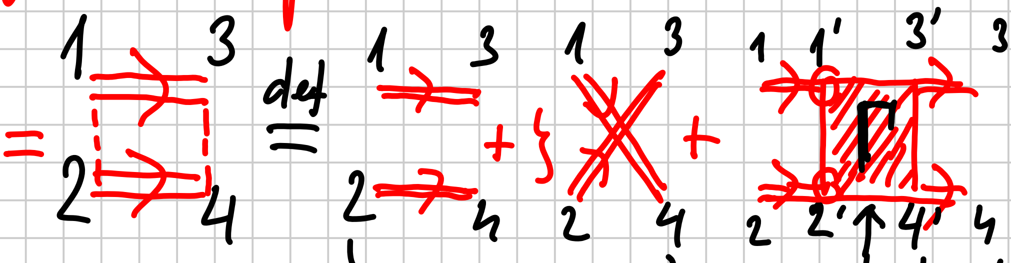
Therefore:

Linked Cluster Theorem

$$\begin{cases} \Omega - \Omega_0 = -\frac{1}{\beta}\sum(\text{all connected graphs}) \\ \Omega_0 = \frac{\zeta}{\beta}\sum_\alpha \ln\left(1-\zeta e^{-\beta(\varepsilon_\alpha-\mu)}\right) \end{cases}$$

Diagrams for 2-particle Green's Function

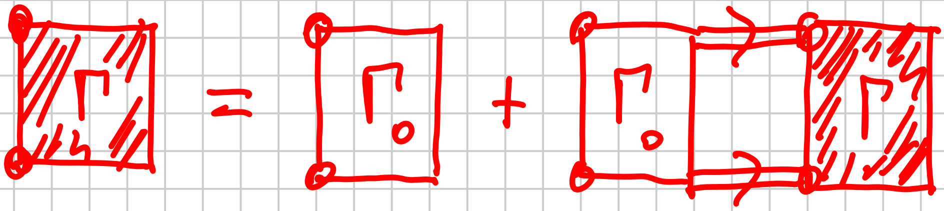
$$G^{II}_{1234} = \langle c_1 c_2 c_3^* c_4^* \rangle_S =$$



For fermions:

$$G^{II}_{1234} = G_{13}\cdot G_{24} - G_{14}G_{23} + \sum_{\{1'2'3'4'\}} G_{11'}G_{22'}\Gamma_{1'2'3'4'}G_{3'3}G_{4'4}$$

Bethe-Salpeter Equation for Γ

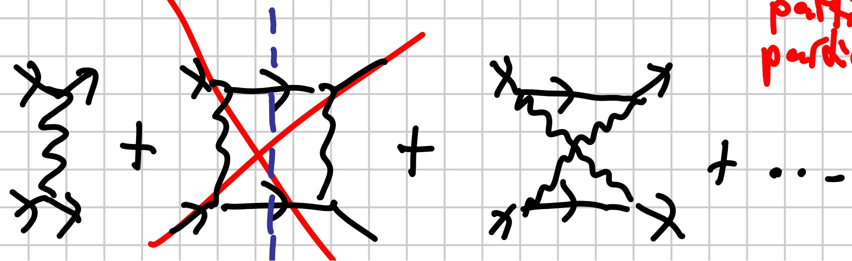


$$\Gamma = \Gamma_0 + \Gamma_0 \otimes G G \otimes \Gamma$$

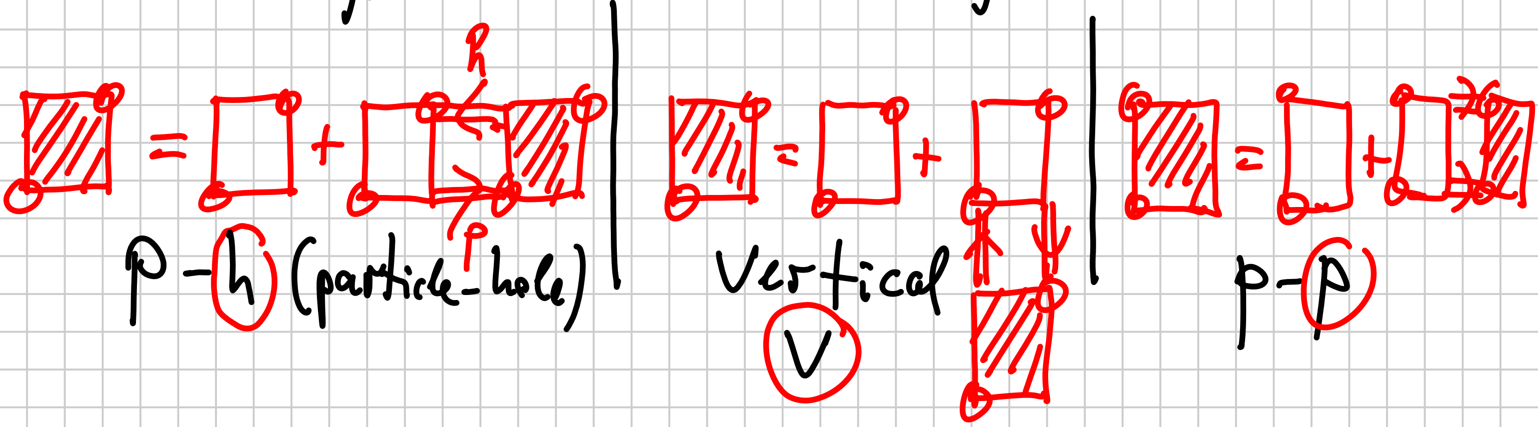
where Γ_0 - irreducible vertex = could not "cut" in 2-line (pp-line)

particle →
particle →

$$\Gamma_0^{pp} =$$



3-different channel of BSE



Parquet Equation: channel: $i = \{h, v, p\}$

BSE: (*) $\Gamma = \Lambda_i + \underbrace{\Lambda_i \otimes G G \otimes \Gamma}_{\Gamma_i = \langle \Lambda_i G G \Gamma \rangle_i} = \underset{\text{irr.}}{\Lambda_i} + \underset{\text{red.}}{\Gamma_i}$ (**)

where $\Lambda_i \equiv \Gamma_0^i$

Introduce: Λ - fully irreducible vertex (FIR)

Then, it is obvious:

$$\Lambda_p = \Lambda + \Gamma_h + \Gamma_v \,;\quad \Lambda_h = \Lambda + \Gamma_v + \Gamma_p \,;\quad \Lambda_v = \Lambda + \Gamma_h + \Gamma_p$$

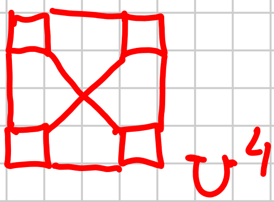
Then from (*) and (**)

$$\begin{cases} \Gamma = \Lambda + \Gamma_p + \Gamma_h + \Gamma_v \\ \Gamma_p = \langle (\Lambda + \Gamma_h + \Gamma_v) G G \Gamma \rangle_p \\ \Gamma_h = \langle (\Lambda + \Gamma_v + \Gamma_p) G G \Gamma \rangle_h \\ \Gamma_v = \langle (\Lambda + \Gamma_p + \Gamma_h) G G \Gamma \rangle_v \end{cases}$$

Parquet Equation

For Λ one can used U^{as}

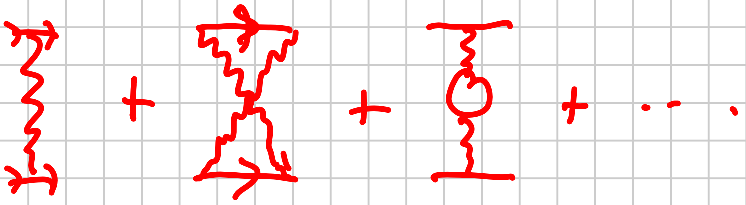
$$\Lambda_{1234} = {}^{1}_{2}\boxed{U}^{3}_{4} + \ldots\; U^4$$



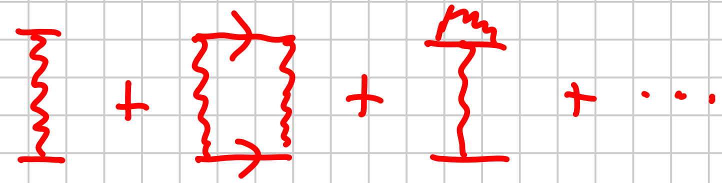
Example: Diagrams for Γ's

Γ =

p: Γ_0^{pp} =



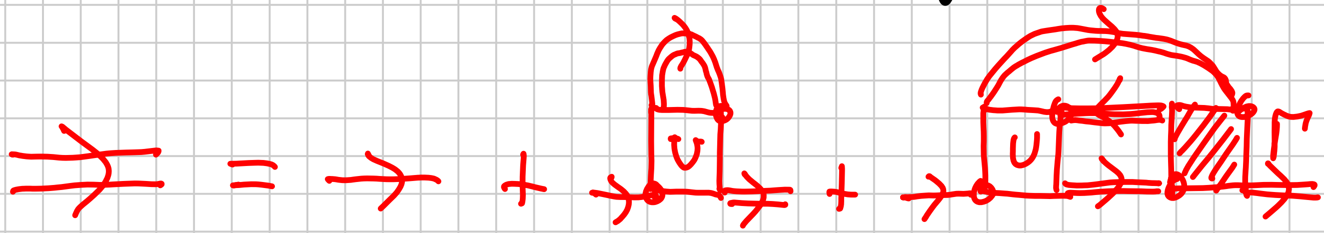
h: P_0^{ph} =



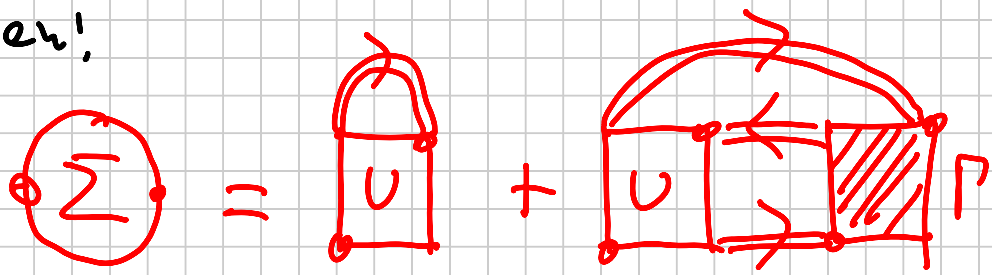
Schwinger-Dyson Equation

(relation between Σ and Γ)

Since we can write "general" diagram for G^I:



then:



$$\Sigma = U \otimes G + U \otimes G G G \otimes \Gamma$$

(Schwinger-Dyson)

Ward-identities

(symmetry relations between G^{I} and G^{II})

We use a simple one field Path Int: $\phi(\underbrace{\vec{r},\tau}_{x_\mu}) \equiv \phi(x_\mu)$, $\mu = 0, \dots, 3$

$$Z[J] = \int D[\phi]\, e^{-S[\phi] + \sum_i \phi_i J_i}$$

$$S[\phi] \underset{\text{?}}{=} -\frac{1}{2}\phi_1 (G^0)^{-1}_{12}\phi_2 + \frac{1}{4}\phi_1\phi_2\Gamma^0_{1234}\phi_3\phi_4$$

Infinitesimal symmetry transformation:

$$\phi_1(\vec{r},\tau) \to \phi_1(\vec{r},\tau) + \underbrace{\varepsilon f_1(\vec{r},\tau)}_{\sum_j \varepsilon_{ij} f_j}$$

(more general)

then:

$$\delta S \overset{\text{def}}{=} -\varepsilon \int d^4x\, j_\mu(x)\partial^\mu f(x)$$

If the fields are classical (φ) then

$$\frac{\delta S[\varphi]}{\delta\varphi(x)} = 0 \qquad \text{stational}$$

and partial integration (with $f(x) \underset{x\to\infty}{\to} 0$) lead to continuity equation:

$$\partial^\mu j_\mu = 0$$

and existence of constant of motion:

$$Q = \int d\vec{r}\, j_0(x) \quad \text{- conserved}$$

additional symmetry of $S[\varphi]$ implies a conservation Law (Noether)

In Quantum case! $\phi_1 \to \phi_1 + \hat{\varepsilon} \ddot{f}[\phi]$

change of $D[\phi]$ with Jacobian:

$$D[\phi] \to D[\phi] \cdot \det\left(\delta_{12} - \varepsilon \frac{\delta f_1}{\delta \phi_2}\right) \approx D[\phi]\left(1 - \varepsilon \frac{\delta f_1}{\delta \phi_1}\right) + O(\varepsilon^2)$$

Symmetry: $Z[J]$ is invariant under transformation

$$Z[J] = \int D[\phi]\left(1 - \varepsilon \frac{\delta f_1[\phi]}{\delta \phi_1}\right) e^{-S[\phi]+\phi J}\left(1 + \frac{\delta S[\phi]}{\delta \phi_1} + J_1\right)\varepsilon f_1[\phi] + O(\varepsilon^2)$$

$$Z = \text{inv.} \Rightarrow \varepsilon \int \dots = 0$$

$$\int D[\phi]\, e^{-S[\phi]+\phi J}\left[\left(\frac{\delta S[\phi]}{\delta \phi_1} + J_1\right) f_1[\phi] - \frac{\delta f_1[\phi]}{\delta \phi_1}\right) = 0$$

or equivalently: $\phi \Rightarrow \frac{\delta}{\delta J}$, then: (Q-Noether)

$$\left[\left(\frac{\delta S[\phi]}{\delta \phi_1} + J_1\right) f_1\left[\frac{\delta}{\delta J}\right] - \frac{\delta f_1\left[\frac{\delta}{\delta J}\right]}{\delta \phi_1}\right) Z[J] = 0$$

if $f_1[\phi]$ = Symmetry of S, then:

$$\frac{\delta S[\phi]}{\delta \phi_1} \cdot f_1[\phi] = 0$$

$\Rightarrow D[\phi]$ = invariant!

$$J_1 f_1\left[\frac{\delta}{\delta J}\right] Z[J] = 0$$

Ward
Takahashi
Pitaevskii

relation between G^{I} and G^{II} or Γ