THÉORIE

DU

MOUVEMENT DE LA LUNE,

PAR CH. DELAUNAY,

MEMBRE DE L'INSTITUT IMPÉRIAL DE FRANCE.

TOME PREMIER.

PARIS,

MALLET-BACHELIER, IMPRIMEUR-LIBRAIRE

DES COMPTES RENDUS HEBDOMADAIRES DES SÉANCES DE L'ACADÉMIE DES SCIENCES,

QUAI DES GRANDS-AUGUSTINS, 55.

1860.

CHAPITRE V.

DÉTAIL DES 57 OPÉRATIONS EFFECTUÉES POUR FAIRE DISPARAITRE LES TERMES LES PLUS IMPORTANTS DE LA FONCTION PERTURBATRICE.

Nous avons expliqué, dans le chapitre III, la marche que nous avons suivie pour faire disparaître successivement de la fonction perturbatrice les divers termes capables de fournir des inégalités d'un ordre inférieur au quatrième. Nous avons dit qu'il nous a fallu, pour cela, effectuer 57 opérations distinctes, dont chacune se traduit en définitive par un changement de variables. Nous nous proposons de donner ici le détail des calculs qui se rapportent à ces diverses opérations, pour chacune desquelles nous n'avons autre chose à faire que d'appliquer une des quatre règles données aux nos **29**, **30** et **31**.

On se rappelle que les équations différentielles qu'il s'agit d'intégrer sont les suivantes :

$$\frac{dL}{dt} = \frac{dR}{dl}, \qquad \frac{dG}{dt} = \frac{dR}{dg}, \qquad \frac{dH}{dt} = \frac{dR}{dh},$$
$$\frac{dl}{dt} = -\frac{dR}{dL}, \qquad \frac{dg}{dt} = -\frac{dR}{dG}, \qquad \frac{dh}{dt} = -\frac{dR}{dH}.$$

La fonction perturbatrice R qui entre dans ces équations, est donnée en fonction du temps t et des éléments variables de la Lune par le développement du no **14** ; mais on peut aussi la prendre dans le chapitre IV, en ayant soin de ne conserver, dans les coefficients des divers termes, que les parties qui ne sont pas accompagnées d'indications en petits chiffres placés au dessous.

Des six variables L, G, H, l, g, h, auxquelles se rapportent les équations différentielles précédentes, il n'y a que les trois dernières qui entrent explicitement dans la valeur de R. Les trois autres L, G, H, y sont remplacées par les éléments a, e, γ, auxquelles elles sont liées par les relations

$$L = \sqrt{a\mu}, \qquad G = L\sqrt{1-e^2}, \qquad H = G(1-2\gamma^2).$$

NUMÉROS des opérations.	ARGUMENTS DES TERMES que ces opérations sont destinées à faire disparaître de R.	ORDRES de ces opérations.
38	$3l$	3e ordre.
39	$2h+2g+5l-2h'-2g'-2l'$	3e ordre.
40	$2h+2g-l-2h'-2g'-2l'$	3e ordre.
41	$2h+2g-2h'-2g'-2l'$	1er ordre.
42	$2h+2g-2h'-2g'-3l'$	2e ordre.
43	$2h+2g-2h'-2g'-l'$	2e ordre.
44	$2h+2g-2h'-2g'-4l'$	3e ordre.
45	$2h+2g-2h'-2g'$	3e ordre.
46	$h+g-h'-g'-l'$	2e ordre.
47	$h+g-h'-g'-2l'$	3e ordre.
48	$h+g-h'-g'$	2e ordre.
49	$2g$	2e ordre.
50	$2h+2g-2h'-2g'-2l'$ (une seconde fois)	3e ordre.
51	$2g$ (une seconde fois)	3e ordre.
52	$2h-2h'-2g'-2l'$	1er ordre.
53	$2h-2h'-2g'-3l'$	2e ordre.
54	$2h-2h'-2g'-l'$	2e ordre.
55	$2h-2h'-2g'-4l'$	3e ordre.
56	$2h-2h'-2g'$	3e ordre.
57	l', $2l'$, $3l'$, $4l'$ (une seconde fois)	3e ordre.

1re OPÉRATION

destinée à faire disparaître les termes (2), (3), (4), (5) *et* (6) *de* R.

Prenons dans R les termes (2), (3), (4), (5) et (6) *, dans lesquels les arguments sont l', $2l'$, $3l'$, $4l'$, $5l'$, et supposons que R se réduise à ces termes seuls, de sorte que l'on ait

$$R = m'\frac{a^2}{a'^3}\left[\frac{3}{4}e' - \frac{9}{2}\gamma^2 e' + \frac{9}{8}e^2 e' + \frac{27}{32}e'^3 + \frac{9}{2}\gamma^4 e' - \frac{27}{4}\gamma^2 e^2 e' - \frac{81}{16}\gamma^2 e'^3 + \frac{81}{64}e^2 e'^3 + \frac{261}{256}e'^5 \right.$$
$$\left. + \frac{27}{4}\gamma^4 e^2 e' + \left(\frac{45}{64}e' - \frac{225}{16}\gamma^2 e' + \frac{225}{64}e^2 e'\right)\frac{a^2}{a'^2}\right]\cos l'$$

De ces valeurs de L, G, H, on déduit

$$\frac{da}{dL} = \frac{1}{an}\left\{2 + \left(\frac{359}{8} - \frac{375}{2}\gamma^2 + \frac{15371}{128}e^2 + \frac{5385}{16}e'^2\right)\frac{n'^4}{n^4} + \left(140 - 396\gamma^2 + \frac{18411}{8}e^2 + 1890\,e'^2\right)\frac{n'^5}{n^5}\right.$$
$$\left. + \frac{41875}{64}\frac{n'^6}{n^6} + \frac{483281}{288}\frac{n'^7}{n^7}\right\},$$

$$\frac{da}{dG} = -\frac{1}{an}\left\{\left(\frac{387}{4} - \frac{1473}{4}\gamma^2 - \frac{22933}{128}e^2 + \frac{5805}{8}e'^2\right)\frac{n'^4}{n^4} + \left(420 - 1476\gamma^2 - \frac{5997}{8}e^2 + 5670\,e'^2\right)\frac{n'^5}{n^5}\right.$$
$$\left. + \frac{126279}{64}\frac{n'^6}{n^6} + \frac{1880475}{288}\frac{n'^7}{n^7}\right\},$$

$$\frac{da}{dH} = -\frac{1}{an}\left\{\left(\frac{33}{4} - \frac{69}{4}\gamma^2 - \frac{771}{8}e^2 + \frac{495}{8}e'^2\right)\frac{n'^4}{n^4} + \left(40 - 120\gamma^2 - 399\,e^2 + 540\,e'^2\right)\frac{n'^5}{n^5}\right.$$
$$\left. + \frac{6191}{32}\frac{n'^6}{n^6} + \frac{94481}{144}\frac{n'^7}{n^7}\right\};$$

$$\frac{de}{dL} = \frac{1}{a^2ne}\left\{1 - e^2 + \left(\frac{927}{32} - \frac{691}{8}\gamma^2 - \frac{36549}{128}e^2 + \frac{13905}{64}e'^2\right)\frac{n'^4}{n^4} + \frac{412}{3}\frac{n'^5}{n^5} + \frac{496405}{768}\frac{n'^6}{n^6}\right\},$$

$$\frac{de}{dG} = -\frac{1}{a^2ne}\left\{1 - \frac{1}{2}e^2 - \frac{1}{8}e^4 - \frac{1}{16}e^6\right.$$
$$\left. + \left(\frac{927}{32} - \frac{691}{8}\gamma^2 - \frac{57693}{128}e^2 + \frac{13905}{64}e'^2\right)\frac{n'^4}{n^4} + \frac{412}{3}\frac{n'^5}{n^5} + \frac{496405}{768}\frac{n'^6}{n^6}\right\},$$

$$\frac{de}{dH} = \frac{1}{a^2ne}\cdot\frac{757}{32}e^2\frac{n'^4}{n^4}.$$

$$\frac{d\gamma}{dL} = -\frac{1}{a^2n\gamma}\cdot\frac{17}{4}\gamma^2\frac{n'^4}{n^4},$$

$$\frac{d\gamma}{dG} = \frac{1}{4a^2n\gamma}\left\{1 - 2\gamma^2 + \frac{1}{2}e^2 - \gamma^2e^2 + \frac{3}{8}e^4 - \frac{3}{4}\gamma^2e^4 + \frac{5}{16}e^6\right.$$
$$\left. + \left(\frac{43}{16} + \frac{879}{8}\gamma^2 - \frac{1427}{64}e^2 + \frac{645}{32}e'^2\right)\frac{n'^4}{n^4} + \frac{61}{6}\frac{n'^5}{n^5} + \frac{8021}{192}\frac{n'^6}{n^6}\right\},$$

$$\frac{d\gamma}{dH} = -\frac{1}{4a^2n\gamma}\left\{1 + \frac{1}{2}e^2 + \frac{3}{8}e^4 + \frac{5}{16}e^6 + \left(\frac{43}{16} - \frac{127}{4}\gamma^2 - \frac{1427}{64}e^2 + \frac{645}{32}e'^2\right)\frac{n'^4}{n^4} + \frac{61}{6}\frac{n'^5}{n^5} + \frac{8021}{192}\frac{n'^6}{n^6}\right\}.$$

Quant aux valeurs de l, $h + g + l$ et h, elles seront fournies par les équations différentielles

$$\frac{dl}{dt} = -\frac{dR}{dL}, \qquad \frac{d(h+g+l)}{dt} = -\frac{dR}{dL} - \frac{dR}{dG} - \frac{dR}{dH}, \qquad \frac{dh}{dt} = -\frac{dR}{dH},$$

qui, en vertu des valeurs de $\frac{da}{dL}$, $\frac{da}{dG}$, $\frac{da}{dH}$, $\frac{de}{dL}$, ..., deviennent

$$\frac{dl}{dt} = -\frac{n'^2}{n}\left[-\frac{225}{2}\gamma^2e' + \frac{225}{4}\gamma^2e^2e' + \left(\frac{675}{16}e' - \frac{243}{2}\gamma^2e' + \frac{2025}{32}e^2e' + \frac{8475}{128}e'^3\right)\frac{n'}{n}\right.$$
$$\left. + \left(\frac{9561}{64}e' - \frac{19197}{32}\gamma^2e' + \frac{63801}{128}e^2e'\right)\frac{n'^2}{n^2} + \frac{448647}{512}e'\frac{n'^3}{n^3} + \frac{9481465}{2048}e'\frac{n'^4}{n^4}\right]\cos l'$$
$$- \frac{n'^2}{n^2}\left[\frac{825}{8}e'^2 - 297\gamma^2e'^2 + \frac{2475}{16}e^2e'^2 + \frac{38937}{128}e'^2\frac{n'}{n} + \frac{2611785}{1024}e'^2\frac{n'^2}{n^2}\right]\cos 2l'$$
$$- \frac{n'^2}{n^2}\cdot\frac{16425}{128}e'^3\cos 3l',$$

$$\frac{d(h+g+l)}{dt} = -\frac{n'^2}{n^2}\left[\frac{81}{4}\gamma^2e' + \frac{2025}{16}e^2e' - \frac{405}{8}\gamma^4e' - \frac{729}{2}\gamma^2e^2e' - \frac{6075}{128}e^4e'\right.$$
$$- \left(\frac{735}{16}e' - \frac{4779}{32}\gamma^2e' - \frac{112509}{128}e^2e' + \frac{28665}{128}e'^3\right)\frac{n'}{n}$$
$$- \left(\frac{3783}{16}e' - \frac{87525}{128}\gamma^2e' - \frac{2962593}{512}e^2e'\right)\frac{n'^2}{n^2}$$
$$\left. - \frac{4571}{4}e'\frac{n'^3}{n^3} - \frac{184357}{48}e'\frac{n'^4}{n^4} + \frac{2475}{64}e'\frac{a^2}{a'^2}\right]\cos l'$$
$$- \frac{n'^2}{n^2}\left[\frac{99}{2}\gamma^2e'^2 + \frac{2475}{8}e^2e'^2 - \left(\frac{6615}{64}e'^2 - \frac{22977}{64}\gamma^2e'^2 - \frac{469503}{256}e^2e'^2\right)\frac{n'}{n}\right.$$
$$\left. - \frac{60203}{128}e'^2\frac{n'^2}{n^2} - \frac{273133}{96}e'^2\frac{n'^3}{n^3}\right]\cos 2l'$$
$$+ \frac{n'^2}{n^2}\cdot\frac{26215}{128}e'^3\cos 3l'.$$

$$\frac{dh}{dt} = \frac{n'^2}{n}\left[-\frac{225}{4}\gamma^2e^2e' + \frac{225}{32}e^4e' + \left(\frac{27}{16}e' - \frac{81}{8}\gamma^2e' - \frac{567}{16}e^2e' + \frac{339}{128}e'^3\right)\frac{n'}{n}\right.$$
$$\left. + \left(\frac{531}{64}e' - \frac{63}{4}\gamma^2e' - \frac{2997}{16}e^2e'\right)\frac{n'^2}{n^2} + \frac{14709}{512}e'\frac{n'^3}{n^3} + \frac{323351}{2048}e'\frac{n'^4}{n^4}\right]\cos l'$$

Cette formule se continue à la page suivante.

THÉORIE

DU

MOUVEMENT DE LA LUNE,

PAR CH. DELAUNAY,

MEMBRE DE L'INSTITUT IMPÉRIAL DE FRANCE.

TOME SECOND.

PARIS,

GAUTHIER-VILLARS, IMPRIMEUR-LIBRAIRE

DES COMPTES RENDUS HEBDOMADAIRES DES SÉANCES DE L'ACADÉMIE DES SCIENCES,

SUCCESSEUR DE MALLET-BACHELIER,

QUAI DES GRANDS-AUGUSTINS, 55.

1867

Another 448 canonical transformations on 700 pages

Main results:

- Moon's longitude: 53 pages
- Moon's latitude: 52 pages

Remaining errors $O(10^{-4})$

THE ASTRONOMICAL JOURNAL VOLUME 76, NUMBER 3 APRIL 1971

Analytical Lunar Ephemeris: Delaunay's Theory

André Deprit, Jacques Henrard, and Arnold Rom
Boeing Scientific Research Laboratories, Seattle, Washington
(Received 1 December 1970)

Delaunay's constants have been substituted into our analytical solution of the main problem of lunar theory. The results are compared with Delaunay's reduced formulas. Corrections are proposed to four terms in the mean motion of the perigee, three in the mean motion of the node, 45 in the reduced expression for the latitude, and 49 in that for the longitude.

Table III. Corrections to Delaunay's solar terms in the longitude V.

No.	D	F	l	l'	m	α	e	γ	e'	Delaunay's numerator	ALE numerator	Delaunay's denominator	ALE denominator
8			1	−1	1		1	4	1	−351	−99		
12			1	1	1		1	4	1	351	99		
13			1	2	4		1		2	−1557809	−1702505		
58		2	−1		2		1	4		−6447	−6402		
58		2	−1		2		3	2		57025	56743		
58		2	−1		4		1	2		149363	74947		
68		2	−3		2		3	2		1231	8792		
89	2				3		2		2	−149497	−132025		
89	2				5				2	−61969	−91129		
95	2			2	3		2		2	−1782049	−1821475		
95	2			2	5				2	−28021	75659		
102	2		1	1	1	2	1		1	−945	−975		
111	2		3	−1	2		3		1	5705	5453		
111	2		3	−1	3		3		1	173819	170687		
118	2		−1		3		1		2	365281	363337		
118	2		−1		4		1		2	19912163	19901957		
124	2		−1	2	2		1	2	2	183	777	16	64
124	2		−1	2	3		1		2	−46561	−46399		
124	2		−1	2	4		1		2	−47974339	−73237483		
151	2	2		1	2		2	2	1	3785	3813		
151	2	2		1	3			2	1	613	617		
167	2	2	−2	−1	2		2	2	1	5255	5043		
169	2	2	−2	1	2		2	2	1	−6355	−6215		
184	2	−2		−1	4			2	1	−13733	−12837		
191	2	−2	1	−1	3		1	2	1	−81	−71	8	16
196	2	−2	2	−1	2		2	2	1	−6197	−7177		
198	2	−2	2	1	2		2	2	1	1881	2021		
205	2	−2	−1	−1	3		1	2	1	−40795	−39339		
221	2	−4		1	2			4	1	−9	9		
227	2	−4	−1		2		1	4		−45	159		
233	4			−1	6				1	54129983	52839679		
236	4			1	4		2		1	−60359	−145415		
237	4			2	4				2	161	201		
237	4			2	5				2	2429	14237	2560	7680
239	4		1		4		3			307749	320549		
240	4		1	−1	3		3		1	140105	136325		
255	4		−1	−2	4		1		2	2056689	2056609		
275	4	2	1		4		1	2		−467	−563		
314	6			1	6				1	2853	−3715		
369	1		−2		3	1	2			−680863	−797863		
372	1		−2	1	2	1	2		1	79689	216189		
399	1	−2		1		1	2	2	1	145	155	48	144
399	1	−2		1	2	1		2	1	25649	26795		
405	1	−2	2	1		1	2	2	1	−115	−395		
418	3		1		4	1	1			1712803	367957	30720	7680
419	3		1	−1	3	1	1		1	6095	25595	768	3072
421	3		1	1	3	1	1		1	−1847	−15181	128	1024
432	3		−2		3	1	2			−2937983	−2931935		
462	5				5	1				3911	1481		

H. Poincaré, Bulletin Astronomique (1908)

MÉMOIRES ET OBSERVATIONS.

SUR LES PETITS DIVISEURS DANS LA THÉORIE DE LA LUNE;

Par H. POINCARÉ.

1. Dans le développement de la théorie de la Lune, on voit s'introduire de petits diviseurs de la forme suivante :

$$p_1 n_1 + p_2 n_2 + p_3 n_3 + p_4 n_4 ;$$

les p sont des entiers, positifs ou négatifs; n_1 et n_2 sont les moyens mouvements de la Lune et du Soleil, n_3 et n_4 sont ceux du périgée et du nœud. Si l'on pose

$$\frac{n_2}{n_1} = m,$$

Mais l'exposant de m est évidemment le même que celui de β. Il peut donc devenir négatif. Si donc on poussait assez loin les développements de Delaunay, on arriverait à des termes où m figurerait à une puissance négative. Mais on n'y arriverait que quand on rencontrerait de très petits diviseurs analytiques, ce qui, nous l'avons dit, ne peut se produire que pour des termes d'ordre très élevé. C'est pour cette raison que cette circonstance a échappé à Delaunay.

Two lessons:

- Canonical transformation are very useful for

  "Classical dynamics in systems with many coupled degrees of freedom"

- Do not worry too much about convergence

GEORG-AUGUST-UNIVERSITÄT
GÖTTINGEN

Real Time Evolution of Quantum Many-Body Systems

Flow Equations and Unitary Perturbation Theory

Stefan Kehrein
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1. Real time evolution in classical mechanics:
   Canonical perturbation theory

2. Real time evolution in quantum mechanics:
   Unitary perturbation theory

3. Unitary perturbation theory in practice:
   Flow equation method

Method

4. Applications (impurity models, local quenches):
   - Spin-boson model
   - Time-dependent ferromagnetic Kondo model

5. Applications (bulk models, global quenches):
   - Quantum quench in a Fermi liquid
   - Quantum quench for 1d fermions with dimerization

6. Outlook

Main collaborators:
A. Mielke (Univ. Heidelberg)
A. Hackl (SAP)
M. Möckel (Cambridge Univ.)
N. Robinson (Oxford Univ.)

2. Real time evolution in classical mechanics: Canonical perturbation theory

"Real time evolution" with small anharmonic terms

$$H = \frac{1}{2}p^2 + \frac{1}{2}q^2 + \frac{g}{4}q^4$$

Ansatz: $q(t) = q^{(0)}(t) + g\,q^{(1)}(t) + O(g^2)$ (initial condition: $q(t=0)=0$)

$$\Rightarrow \quad q^{(0)}(t) = c\,\sin t$$

$$\Rightarrow \quad \ddot{q}^{(1)}(t) = -q^{(1)}(t) - c^3 \sin^3 t$$

$$\Rightarrow \quad q^{(1)}(t) = \frac{c^3}{8}\left(-\frac{5}{4}\sin^3 t - \frac{3}{4}\cos^2 t\,\sin t + 3t\,\cos t\right)$$

"Secular term"

Secular terms invalidate naive perturbation theory for large times!

Much better … Canonical perturbation theory

Find canonical transformation (q,p) → (Q,P) that brings H to normal form:

$$H(q,p) = \frac{1}{2}p^2 + \frac{1}{2}q^2 + \frac{g}{4}q^4$$

$$\tilde{H}(Q,P) = H_0 + \frac{3}{8}\,g\,H_0^2 + O(g^2) \quad \text{with} \quad H_0 = \frac{1}{2}P^2 + \frac{1}{2}Q^2$$

vanishing Poisson bracket → exact solution of dynamics for Q(t), P(t)
($H_0 = E_0$ conserved) possible (no secular terms):

$$Q(t) = Q_0\,\sin(\omega t + \gamma_0) \quad \text{with} \quad \boxed{\omega = 1 + \frac{3}{4}g\,E_0}$$

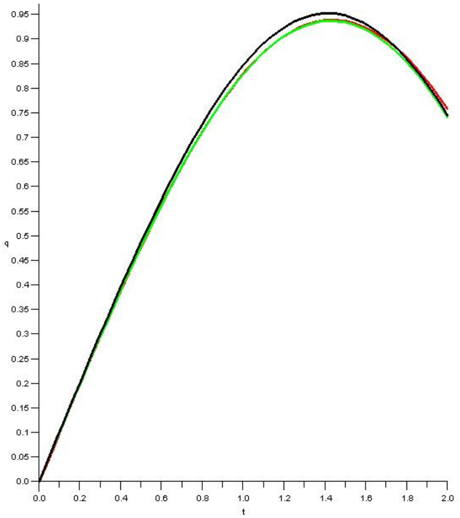
Generating function: $F_2(q,P) = qP + g\left(\frac{5}{32}\,P\,q^3 + \frac{3}{32}\,P^3\,q\right)$

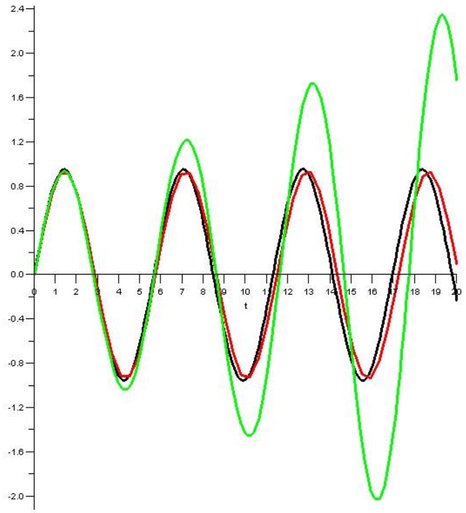
$$Q = \frac{\partial F_2}{\partial P} = q + g\left(\frac{5}{32}\,q^3 + \frac{9}{32}\,P^2\,q\right) \quad , \quad p = \frac{\partial F_2}{\partial q} = \dots$$

Insert solution Q(t), P(t) and reexpress in terms of q(t), p(t):

$$q(t) = Q_0\sin(\omega t + \gamma_0) - g\left(\frac{5}{32}Q_0^3\sin^3(\omega t + \gamma_0) + \frac{3}{32}Q_0^3\cos^2(\omega t + \gamma_0)\sin(\omega t + \gamma_0)\right) + O(g^2)$$

also calculation in first order in g





red line: exact

green line: naive pert. theory

black line: canonical pert. theory

Problem of naive perturbation theory:
naive expansion in coupling constant
produces secular terms

$$\sin(\omega t) = \sin\left((1+\frac{3}{4}g E_0)t\right)$$
$$= \sin t + \frac{3}{4} g E_0 t \cos t + O(g^2)$$

canoncial perturbation theory   naive perturbation theory in coupling

$\Rightarrow$ Same recipe for dynamics of quantum systems:

Perturbation theory based on unitary transformations instead of “naive” perturbation expansion

2. Real time evolution in quantum mechanics: Unitary perturbation theory

U: unitary transformation that diagonalizes the Hamiltonian (approximately)

$$\begin{tikzcd}
H, O, |\Psi_i\rangle \arrow[r, "\mathrm{U}"] \arrow[d, dashed, "\text{Non-perturbative solution of Heisenberg equations of motion for operator O(t)}"] & \tilde{H}, \tilde{O}, |\tilde{\Psi}_i\rangle \arrow[d, "\text{Time evolution}"] \\
O(t), |\Psi_i\rangle & \tilde{O}(t) = e^{i\tilde{H}t}\tilde{O}e^{-i\tilde{H}t}, |\tilde{\Psi}_i\rangle \arrow[l, "\mathrm{U}^\dagger"']
\end{tikzcd}$$

No secular terms!

"Forward-backward transformation"

3. Unitary perturbation in practice: Flow equations

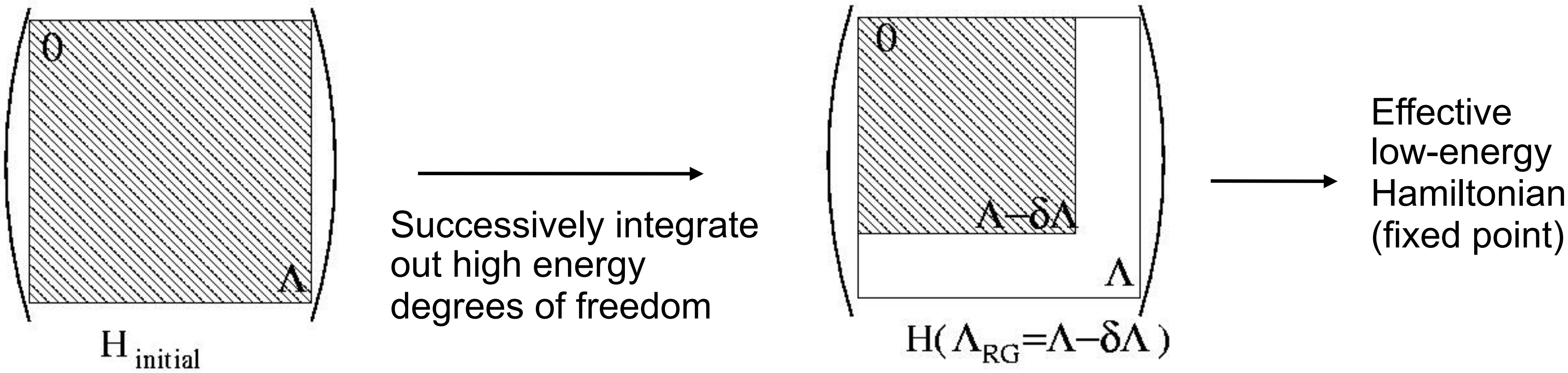
Problem: How to construct U for non-integrable models with a continuum of energy scales?

Wanted: Perturbative method for finding U

Stable expansion for systems with very different energy scales

→ Energy-scale separation

→ Scaling theory / Renormalization theory (K.G. Wilson)



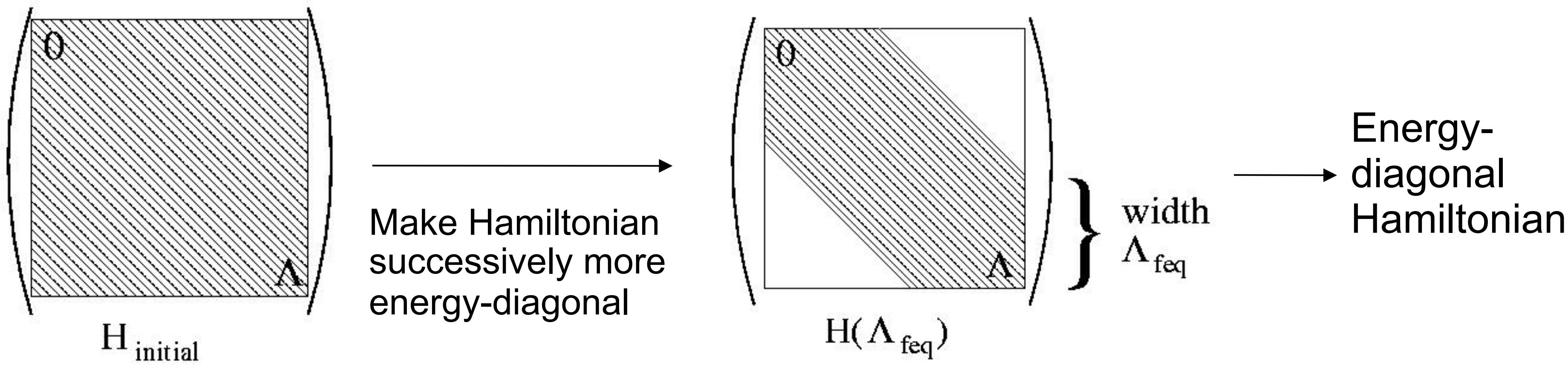
Problem:

Information on full Hilbert space required to construct unitary transformation for forward-backward transformation

Flow Equation Method

F. Wegner (1994)

S. K., The Flow Equation Approach to Many-Body Problems (Springer 2006)



Implementation of flow: Sequence of infinitesimal unitary transformations

One-parameter family of unitarily equivalent Hamiltonians generated by solving the differential equation ("flow equations")

$$\frac{dH}{dB} = [\eta(B), H(B)]$$

with H(B=0) the initial Hamiltonian and an anti-hermitean generator η(B).

Canonical choice of generator (Wegner 1994):

$$H(B) = H_0(B) \text{ [diagonal part] } + H_{int}(B) \text{ [interaction part]}$$

→ define anti-hermitean generator $\eta(B) = [H_0(B), H_{int}(B)]$

$$\boxed{\Rightarrow \quad \lim_{B\to\infty} \eta(B) = 0}$$

→ generates band-diagonal Hamiltonians H(B) with $B^{-1/2} = \Lambda_{feq}$

Challenge: Generation of higher and higher order interaction terms
→ need suitable expansion parameter (typically running coupling)

Advantages:

- RG-like analytical method
- Controlled solution in certain strong-coupling problems (e.g., Kondo model)
- Keeps all states in Hilbert space
  → correlation functions on all energy scales
  → important in non-equilibrium (real time evolution, steady states)

- Avoids secular terms in real time evolution!

4. Applications (impurity models): Spin-boson model

Paradigmatic 2-state system for dissipative quantum mechanics:

$$H = -\frac{\Delta}{2}\sigma_x + \underbrace{\sigma_z \sum_k \lambda_k (b_k + b_k^\dagger)}_{H_{\rm int}} + \sum_k \omega_k b_k^\dagger b_k$$

Environment degrees of freedom → Decoherence and dissipation

Flow equation diagonalization:

$$\frac{d\lambda_k(B)}{dB} = -(\omega_k - \Delta)^2 \lambda_k$$

Energy scale separation

$$\frac{d\Delta(B)}{dB} = -\Delta \sum_k \lambda_k^2 \frac{\omega_k - \Delta}{\omega_k + \Delta} \coth(\beta\omega_k/2)$$

Non-perturbative energy scale (frequency)

$$\Rightarrow \quad H(B=\infty) = -\frac{\Delta_r}{2}\sigma_x + \sum_k \omega_k b_k^\dagger b_k + O(\lambda_k^2) \qquad \Delta_r \propto \left(\frac{\Delta}{\omega_c}\right)^{\alpha/1-\alpha}$$

Time evolution with respect to H(B=∞) is trivial

→ Where is dissipation/decoherence?

→ Transformation of observables!

Transformation of Observables

Transformation of observable O to diagonal basis ("Forward" transformation):

$$\frac{dO(B)}{dB} = [\eta(B), O(B)] \quad , \quad O(B=0) = O$$

Ansatz for σ_z :

$$\sigma_z(B) = h(B)\,\sigma_z + \underbrace{\sigma_x \sum_k \chi_k(B)\,(b_k^\dagger + b_k)}\, + \text{ higher order terms}$$

$$h(B) \xrightarrow{B\to\infty} 0$$

$0 \longrightarrow$ Observable becomes completely entangled with environment degrees of freedom

Decoherence in "conventional" framework:	Decoherence in flow equation framework:
System degrees of freedom become entangled with environment	Hamiltonian diagonal, therefore system observables become entangled with environment

Equilibrium Dynamics (T=0)

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\underset{\displaystyle H,\ O,\ |\Psi_i\rangle}{\mathrm{B}=0} \arrow[r, "\mathrm{U}"] \arrow[d, dashed, "\substack{\text{Non-perturbative solution}\\ \text{of Heisenberg equations of}\\ \text{motion for operator O(t)}}"] & \underset{\displaystyle \tilde{H},\ \tilde{O},\ |\tilde{\Psi}_i\rangle}{\mathrm{B}=\infty} \arrow[d, "\text{Time evolution}"] \\
O(t),\ |\Psi_i\rangle & \tilde{O}(t) = e^{i\tilde{H}t}\tilde{O}e^{-i\tilde{H}t},\ |\tilde{\Psi}_i\rangle \arrow[l, "\mathrm{U}^{\dagger}"']
\end{tikzcd}
$$

Equilibrium: $|\ \tilde{\psi}_i >$ is trivially given as ground state |GS> of $\tilde{\mathrm{H}}$

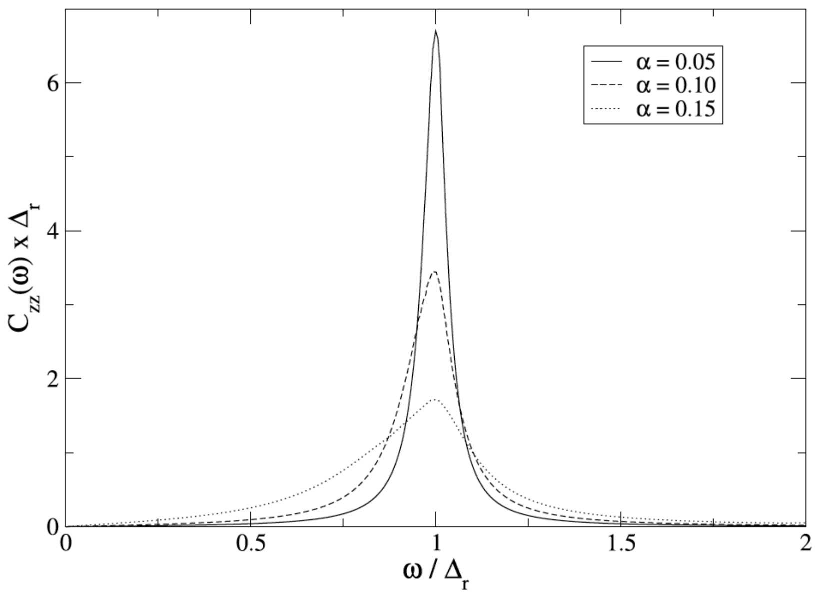
Spin operator in diagonal basis:

$$\sigma_z(B=\infty) = \sigma_x \sum_k \chi_k(B=\infty)\,(b_k^\dagger + b_k)$$

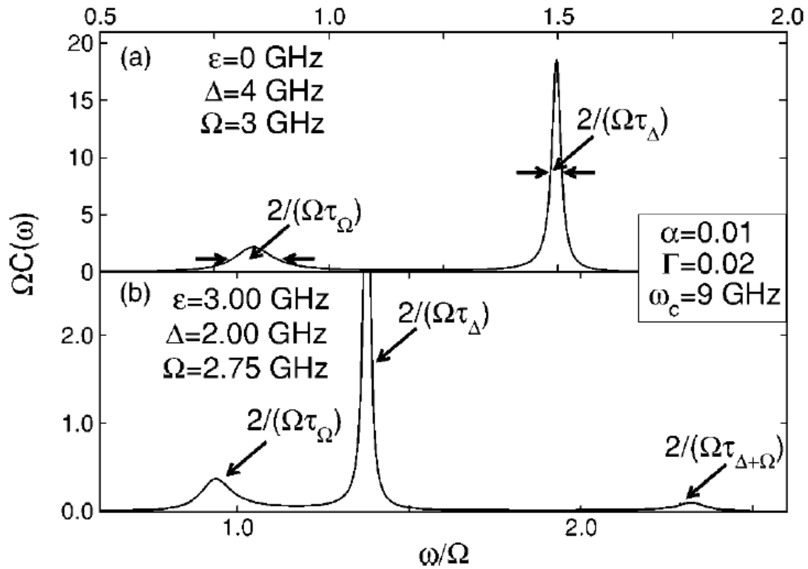
$\rightarrow$ Incoherent spin dynamics:

$$C_{zz}(t) = \frac{1}{2}\langle GS|\left\{\sigma_z(B=\infty)\,,\, e^{iH(B=\infty)t}\sigma_z(B=\infty)e^{-iH(B=\infty)t}\right\}|GS\rangle$$

Practical evaluation: Numerical evaluation of O(10^3) ordinary differential equations (qualitative behavior: analytical calculation!)



Ohmic bath



Structured bath (coupling to bath via harmonic oscillator Ω)

[S. Kleff, S. K., J. von Delft, Phys. Rev. B 70 2004]

Real time evolution

A. Hackl and S. K., Phys. Rev. B 78 (2008), J. Phys. C 21 (2009)

Forward-backward transformation (numerical implementation):

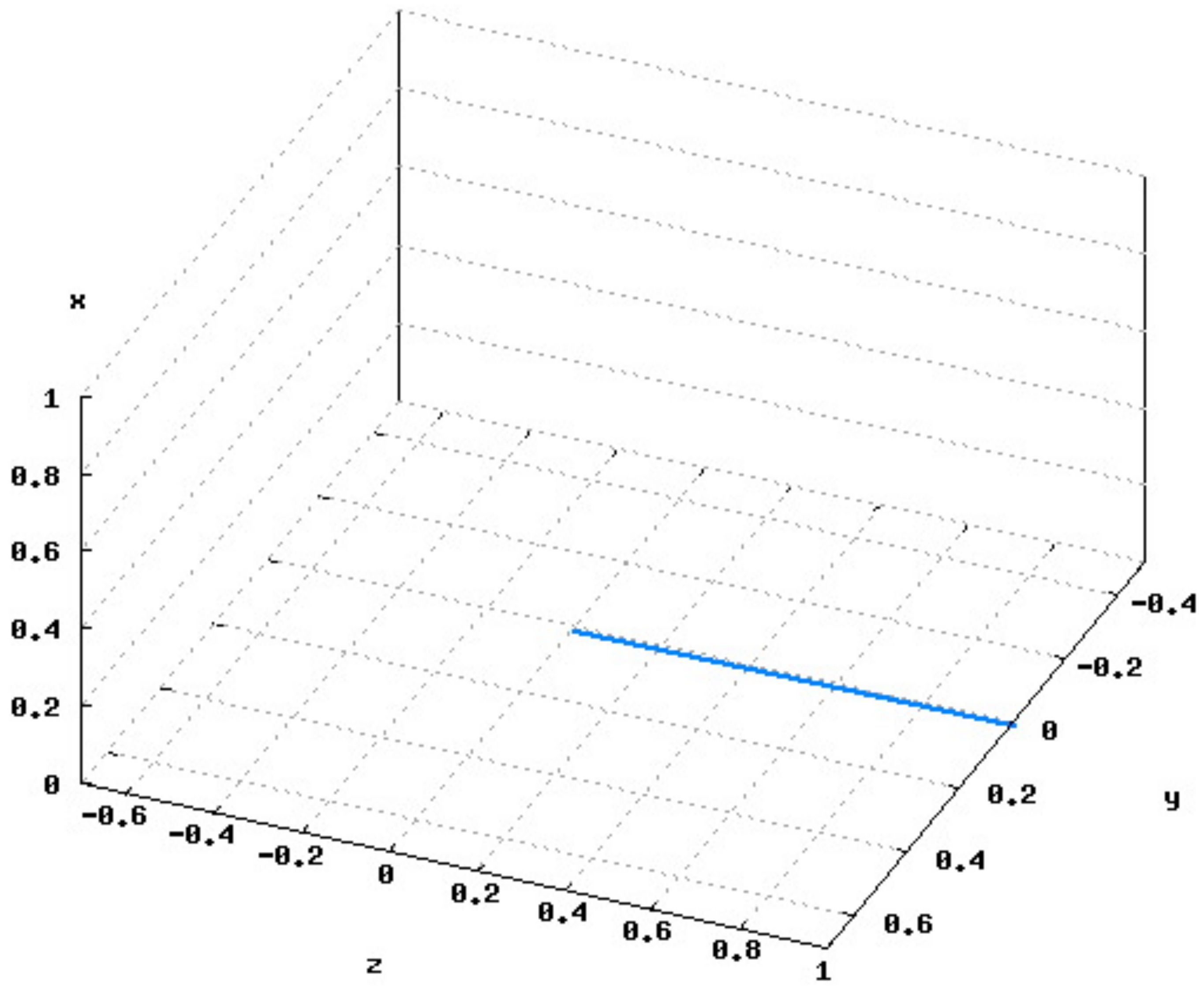
$$\begin{aligned}\sigma_z(t) = {} & z(t)\sigma_z + y(t)\sigma_y \\ & + i\sigma_x \sum \alpha_k(t)\,(b_k - b_k^\dagger) + \sigma_x \sum \beta_k(t)\,(b_k + b_k^\dagger) \\ & + \text{higher order terms}\end{aligned}$$

Evaluate for arbitrary initial state, e.g. here:

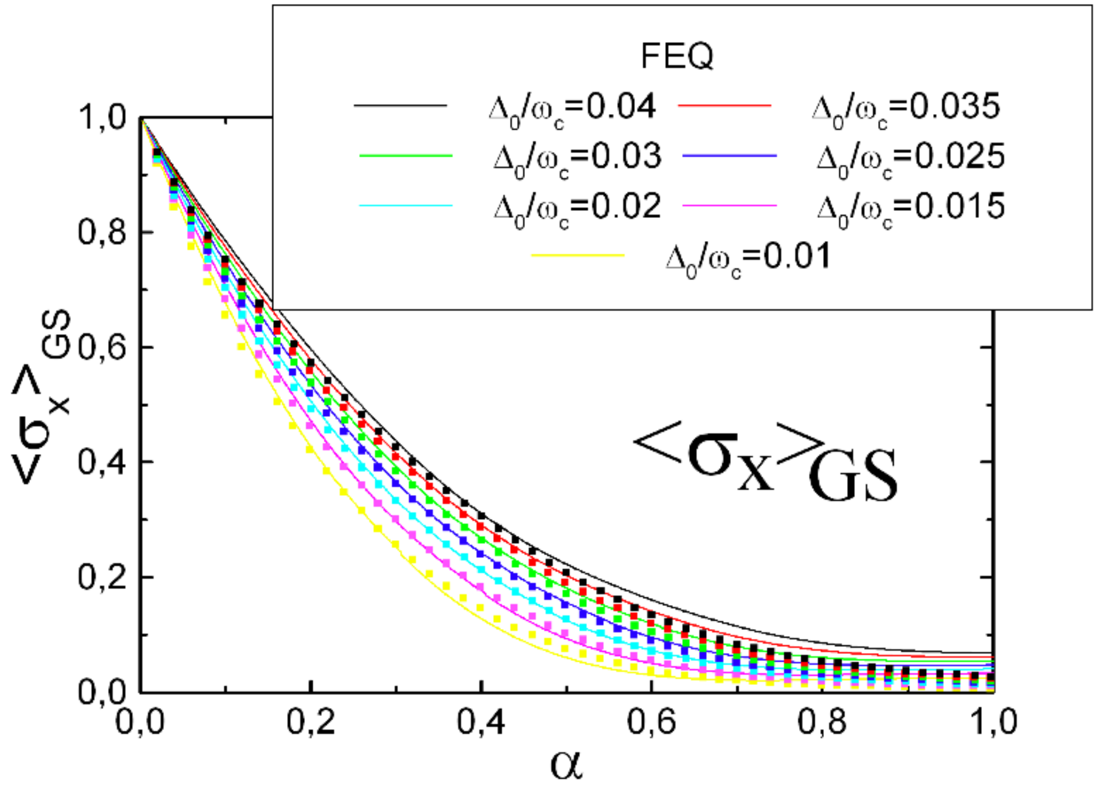
Initial state: $|\Psi_i\rangle = |\uparrow, \text{ relaxed bath}\rangle$

Expectation values: $E_z(t) = \langle \Psi_i | \sigma_z(t) | \Psi_i \rangle = z(t)$

Shows exponential decay (possibly with oscillations) already in this order of flow equation calculation


x
1
0.8
0.6
0.4
0.2
0
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
z
-0.4
-0.2
0
0.2
0.4
0.6
y

Stable long time asymptotics (no secular terms):



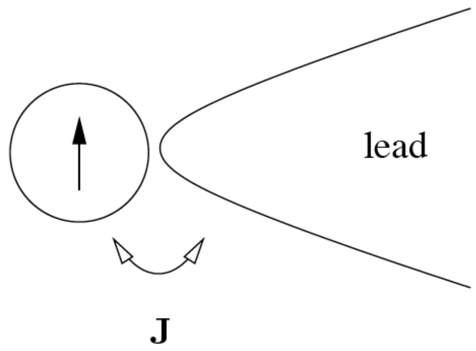
Comparison with NRG data from Costi et al., PRA 68 (2003)

Applications (impurity models): Ferromagnetic Kondo model

A. Hackl, D. Roosen, S. K., W. Hofstetter, Phys. Rev. Lett. 102, 196601 (2009)
A. Hackl, M. Vojta and S. K., Phys. Rev. B 80, 195117 (2009)

$$H_i = \sum_{k,\sigma} \varepsilon_k c^\dagger_{k\alpha} c_{k\alpha} - 0^+ S_z$$

infinitesimal magnetic field



$\Rightarrow$ Product initial state: $|\Psi_i\rangle = |\uparrow\rangle \otimes |FS\rangle$

$$H_f = \sum_{k,\sigma} \varepsilon_k c^\dagger_{k\alpha} c_{k\alpha} - 0^+ S_z + J\vec{S} \cdot \sum c_{k'\alpha} \vec{\sigma}_{\alpha\beta} c_{k\beta}$$

ferromagnetic coupling (J<0):

Coupling constant flows to zero
$\Rightarrow$ Expansion becomes better (asymptotically exact) for long times

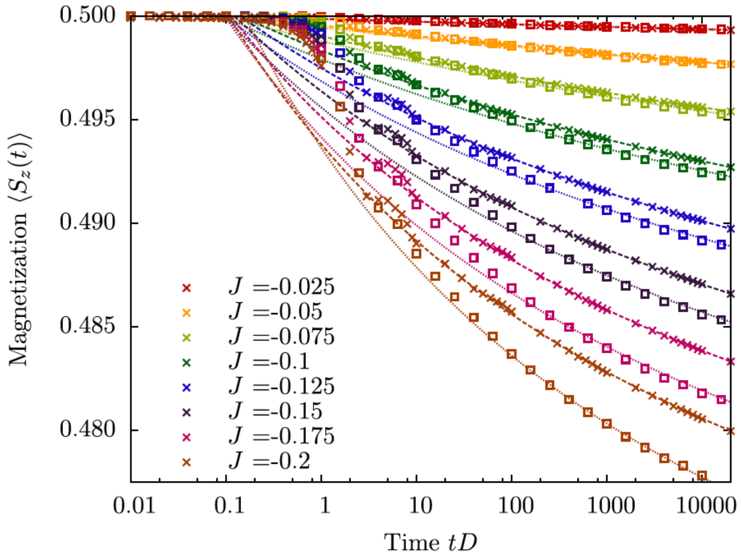
Nonequilibrium spin expectation value (large times):

$$\langle S_z(t)\rangle = \frac{1}{2}\left(\frac{1}{\ln(t) - \frac{1}{\rho J}} + 1 + \rho J + O(J^2)\right).$$

Equilibrium:

$$\langle S_z\rangle_{eq} = \frac{1}{2}\left(1 + \frac{\rho J}{2} + O(J^2)\right)$$

Comparison with TD-NRG:
[Hackl et al., PRL 102 (2009)]



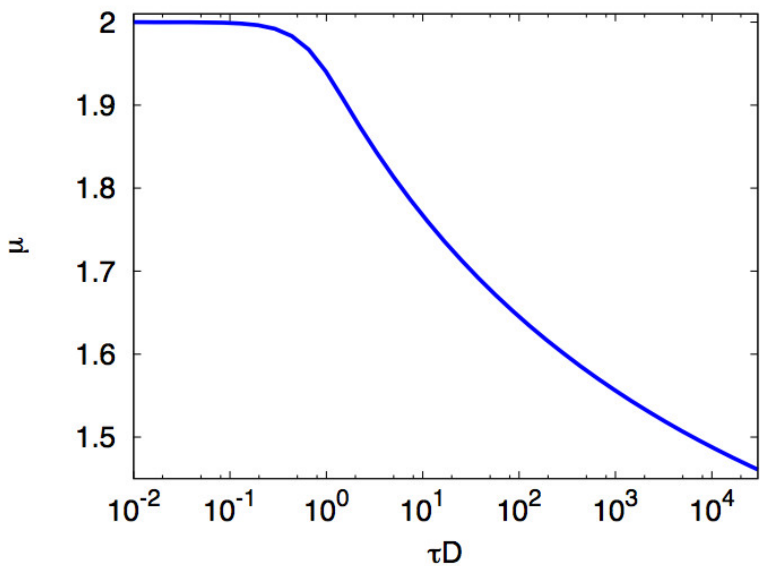
Crossover from adiabatic to instantaneous quenching:
(C. Tomaras, S. K., Eur. Phys. Lett. 93, 47011 (2011))

Coupling J switched on on timescale **τ**

Measure of non-adiabacity:

$$\boxed{\mu \stackrel{def}{=} \frac{\lim_{t\to\infty}\langle O(t)\rangle_{neq} - \langle O\rangle_0}{\langle O\rangle_{eq} - \langle O\rangle_0}}$$

$\Longrightarrow$ Crossover timescale nonperturbative (exponentially large) due to RG flow

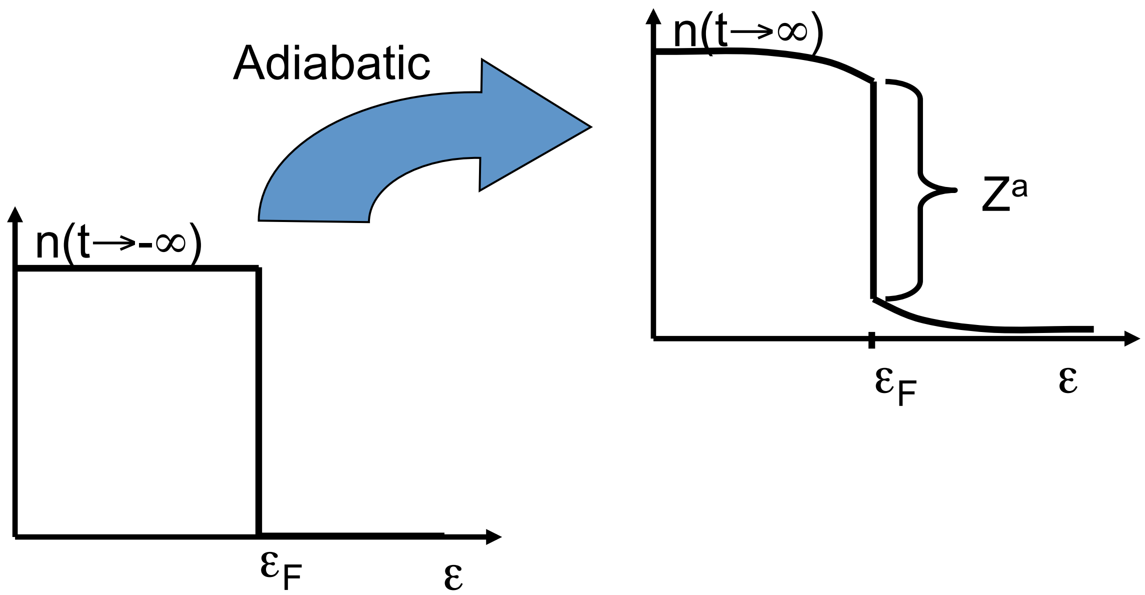


5. Applications (bulk models): Quantum quench in a Fermi liquid

Landau Fermi liquid theory:

Adiabatic switching on of interaction

→ 1 to 1 correspondence between physical electrons and quasiparticles



What happens for sudden switching (global quantum quench)?

Translation-invariant closed system + nonzero excitation energy density

⇒ Thermalization?

M. Moeckel and S. K., Phys. Rev. Lett. 100 (2008), Ann. Phys. 324 (2009)

Hubbard model in d>1 dimensions

$$H = \sum_{k,\alpha} \varepsilon_k c^\dagger_{k\alpha} c_{k\alpha} + U\,\Theta(t) \sum_i \left(n_{i\uparrow} - \frac{1}{2}\right)\left(n_{i\downarrow} - \frac{1}{2}\right)$$

Forward transformation:

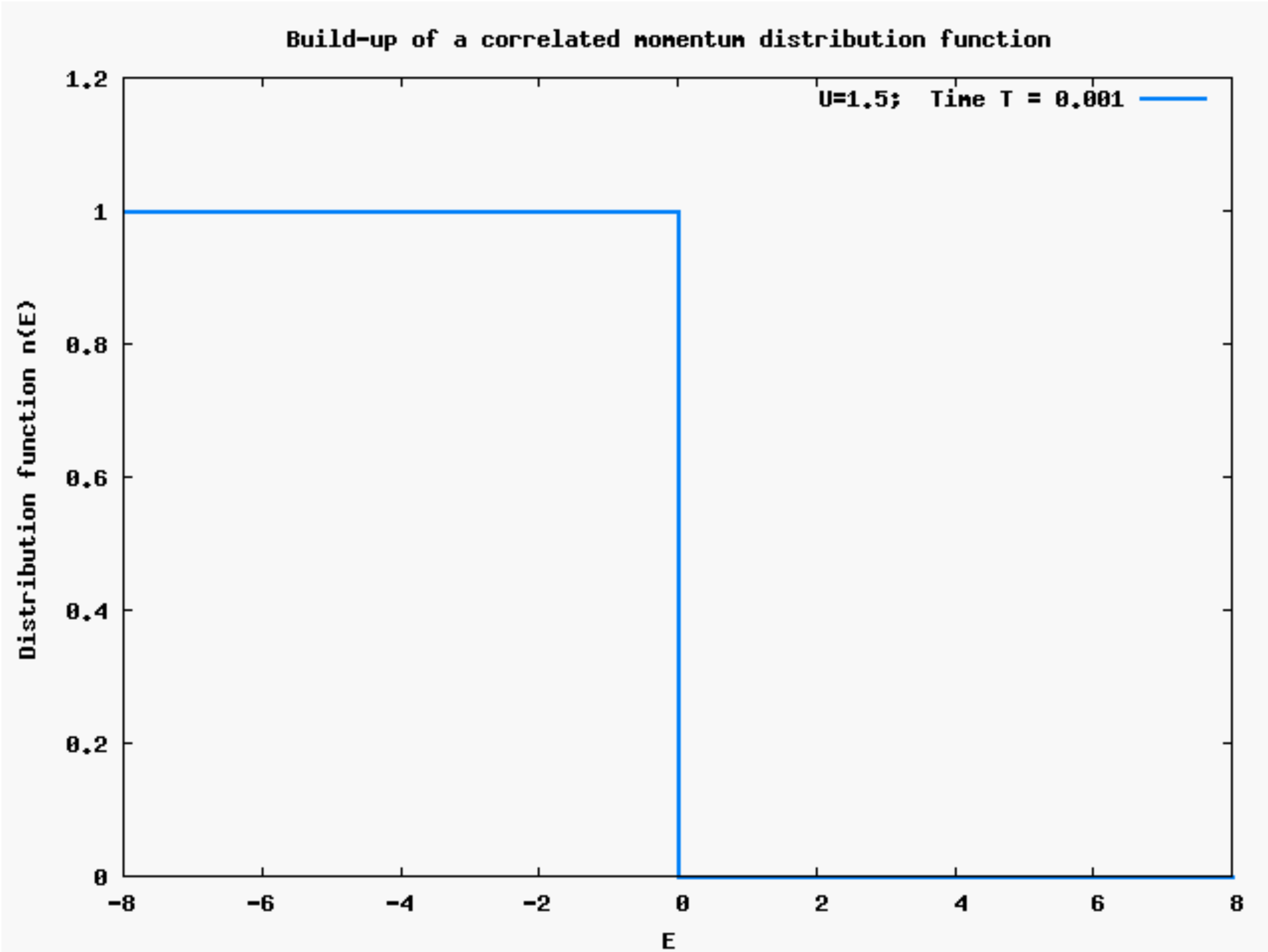
$$c^\dagger_{k\uparrow}(B=\infty) = h_k c^\dagger_{k\uparrow} + \sum_{k'_1,k'_2,k_1} M^k_{k'_1 k'_2 k_1} : c^\dagger_{k'_1\uparrow} c^\dagger_{k'_2\downarrow} c_{k_1\downarrow} : + \text{higher order terms}$$

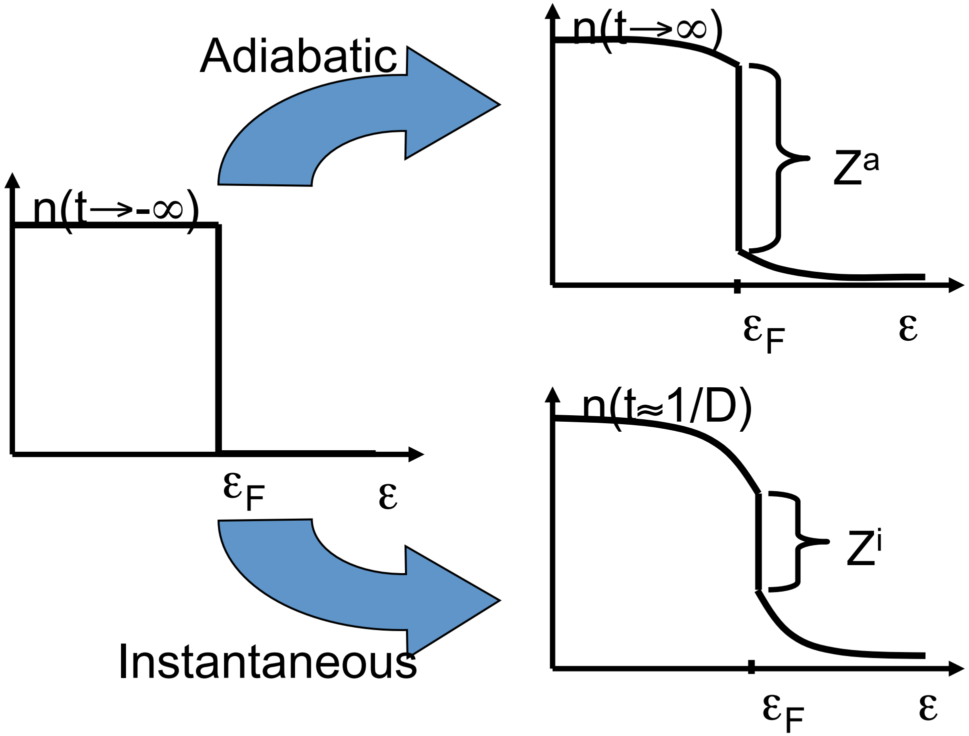
- h_k only nonzero at Fermi surface for zero temperature
- Quasiparticle residue (equilibrium)

$$Z = h^2_{k_F}$$

Real time evolution

Analytical evaluation (with numerical integration) up to order U^2





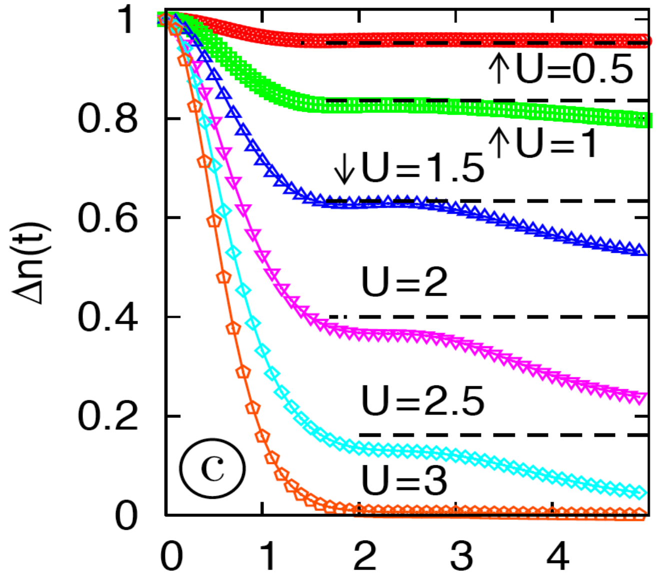
Sudden switching looks like T=0 Fermi liquid with “wrong” quasi-particle residue on time scale $t \propto D^{-1}$

→ Novel metastable <u>prethermalized state</u>

→ Thermalization via QBE on time scale $t \propto U^{-4}$

Numerical confirmation:

Non-equilibrium DMFT with real time QMC
M. Eckstein et al., Phys. Rev. Lett. 103 (2009)



5. Applications (bulk models): 1d fermions with dimerization

F. Essler, S. K., S. Manmana and N. Robinson, arXiv:1311.4557, to appear in PRB

$$H(\delta, U) = -J \sum_l \left(1 + (-1)^l \delta\right) \left(c_l^\dagger c_{l+1} + \text{h.c.}\right) + U \sum_l c_l^\dagger c_l c_{l+1}^\dagger c_{l+1}$$

H(δ,U=0): Peierls insulator, exactly solvable via Bogoliubov transformation,
exactly solved by flow equations

H(δ≠0,U≠0): Non-integrable model

Goal: Thermalization for global quenches $H(\delta_i, U_i=0) \rightarrow H(\delta_f, U_f \neq 0)$?
(Here: tuneable integrability breaking $U_f \neq 0$ for fixed quench amplitude $\delta_i \rightarrow \delta_f$)

Flow equation calculation:
- Up to terms $O(U^2)$
- Analytical calculation (with numerical integration)
- 2- and 4-point functions
- comparison with t-DMRG

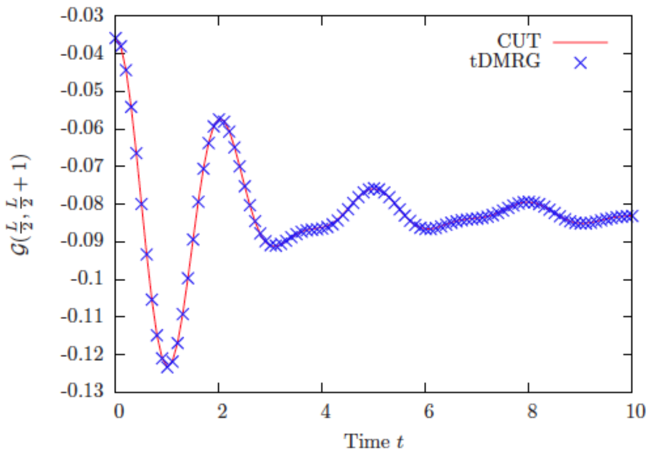


FIG. 10. Comparison of the CUT and t-DMRG results for $\mathcal{G}(L/2, L/2+1) = \langle c_{L/2} c^{\dagger}_{L/2+1} \rangle$ for the quench $\delta_i = 0.75 \to \delta = 0.5$ and $U_i = 0 \to U = 0.15$ on a $L = 50$ chain. The

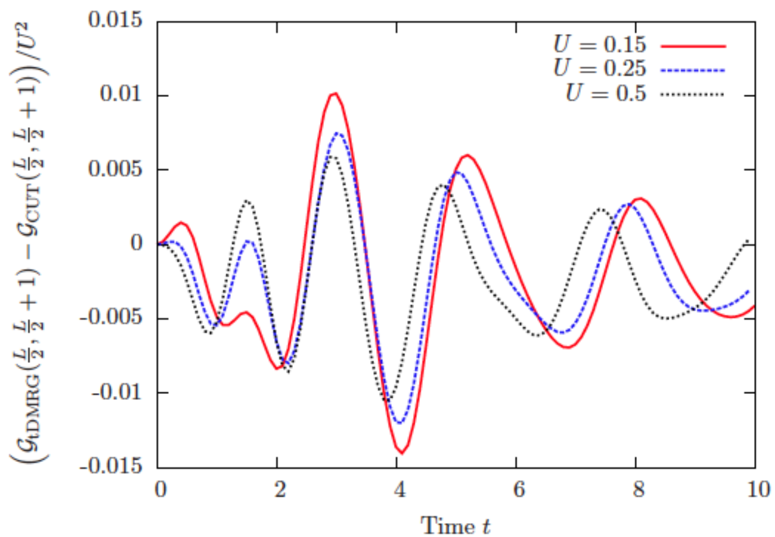


FIG. 13. Rescaled difference between the t-DMRG and CUT data for $\mathcal{G}(25, 26)$ and different values of U.

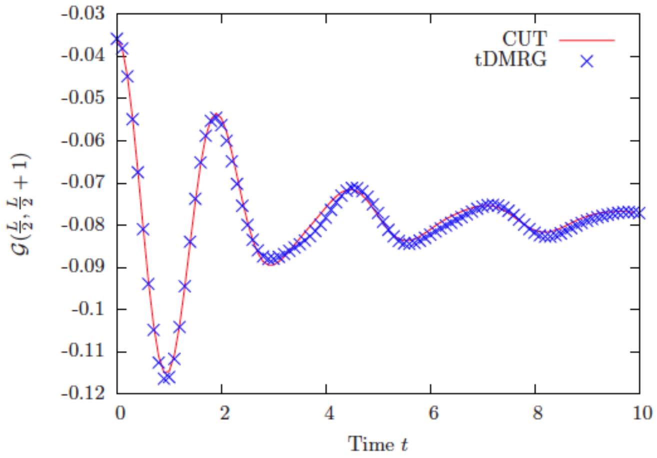


FIG. 12. Comparison of the CUT and t-DMRG results for $\mathcal{G}(L/2, L/2+1) = \langle c_{L/2} c^{\dagger}_{L/2+1} \rangle$ for the quench $\delta_i = 0.75 \to \delta = 0.5$ and $U_i = 0 \to U = 0.5$ on a $L = 50$ chain.

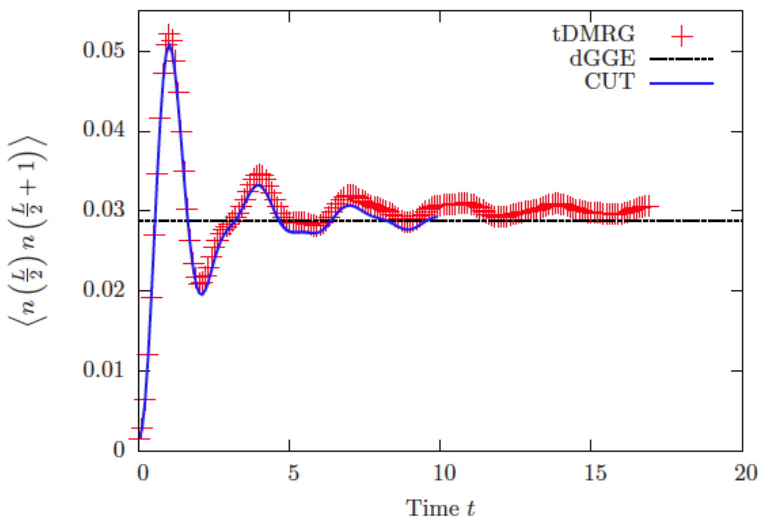


FIG. 18. Nearest neighbour density-density correlation function $\langle n(\frac{L}{2}) n(\frac{L}{2}+1) \rangle$ for a quench from $\delta_i = 0.8 \to \delta_f = 0.4$ and $U = 0 \to 0.4$ computed by t-DMRG for system size

6. Outlook

Unitary perturbation theory based on flow equations:

- Perturbative method in the sense of weak coupling RG
- No secular terms in time evolution
- Gives exponential/power-law/etc. decays in lowest order calculations
- How to incorporate Boltzmann equation dynamics?
- Convergence properties of observable transformation?