Winter term 2012/13

Exercise Sheet 9, Theoretical Quantum and Atom Optics

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To be returned on Tuesday, 08/01/2013, in the tutorials

Exercise 18. Bogoliubov Equations

Start from the time-dependent Gross-Pitaevskii equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r},t)+V(\mathbf{r})\psi(\mathbf{r},t)+U_0|\psi(\mathbf{r},t)|^2\psi(\mathbf{r},t)=i\hbar\frac{\partial\psi(\mathbf{r},t)}{\partial t}, \tag{1}$$

and assume that it has an unperturbed stationary solution $\psi_0(\mathbf{r},t)=\phi_0(\mathbf{r})e^{-i\mu t/\hbar}$.

(i) Derive the linearized equation for a perturbation $\delta\psi(\mathbf{r},t)$ around this solution. To do so, assume that the wavefunction of the condensate only slightly deviates from the unperturbed state, $\psi(\mathbf{r},t)=\psi_0(\mathbf{r},t)+\delta\psi(\mathbf{r},t)$, and linearize the GP equation in terms of this small deviation.

(ii) Search for the solutions $\delta\psi(\mathbf{r},t)$ of the linearized equation with the Bogoliubov Ansatz

$$\delta\psi(\mathbf{r},t)=e^{-i\mu t/\hbar}[u(\mathbf{r})e^{-i\omega t}-v^*(\mathbf{r})e^{i\omega^* t}].$$

Derive the Bogoliubov equations for $u(\mathbf{r}),v(\mathbf{r})$.

(iii) Assume a uniform Bose gas, $V(\mathbf{r})=0$, and its unperturbed homogeneous solution $\psi(\mathbf{r},t)=\sqrt{n(\mathbf{r})}e^{-i\mu t/\hbar}$, where the chemical potential is given by $\mu=nU_0$.

Solutions of the Bogoliubov equations can now be sought in the form of plane waves $u(\mathbf{r})=u_q\frac{e^{i\mathbf{qr}}}{\sqrt{\mathcal{V}}}$, $v(\mathbf{r})=v_q\frac{e^{i\mathbf{qr}}}{\sqrt{\mathcal{V}}}$, where $\mathcal{V}$ is the volume of the system and u_q,v_q are constant coefficients. Show that the Bogoliubov equations can then be written in the form

$$\left(\frac{\hbar^2q^2}{2m}+nU_0-\hbar\omega\right)u_q-nU_0v_q=0 \tag{2}$$

$$\left(\frac{\hbar^2q^2}{2m}+nU_0+\hbar\omega\right)v_q-nU_0u_q=0 \tag{3}$$

Find the dispersion relation $\omega(q)$ by studying the solvability condition for this homogeneous linear system. Show that $\omega(q)$ is asymptotically linear for small q and quadratic for large q. Convince yourself that for attractive interactions, $U_0<0$, the Bogoliubov spectrum contains imaginary modes, signalling instability towards collapse.

5 Points

Exercise 19. Dark solitons

The repulsive ($U_0>0$), uniform ($V(x)=0$), one-dimensional Gross-Pitaevskii equation posseses the solitonic solution

$$\psi(x,t)=\sqrt{n_0}\left[i\frac{u}{s}+\sqrt{1-u^2/s^2}\tanh\left(\frac{x-ut}{\sqrt{2}\xi_u}\right)\right]e^{-i\mu t/\hbar}, \tag{4}$$

where $\xi=\hbar/\sqrt{2mn_0U_0}$ is the so-called coherence length, $\xi_u=\xi/\sqrt{1-u^2/s^2}$, and $s=\sqrt{\frac{n_0U_0}{m}}$ is the sound velocity in the uniform gas with density n_0.

The parameter $u<s$ denotes the soliton velocity.

(i) Find the minimum density n_{min} of this object.

(ii) Show that the soliton velocity is equal to the bulk sound velocity evaluated at the density n_{min}, i.e., to the sound velocity of a uniform gas with density n_{min}.

(iii) For simplicity, focus on the case $u>0$ now. Show that the total phase change $\varphi(x\to+\infty)-\varphi(x\to-\infty)$ (where $\varphi\in(-\pi,\pi]$ denotes the argument of ψ) across the soliton at fixed time is monotonically related to its velocity and is equal to $-2\arccos(u/s)$.

5 Points