WS 2012/2013

Exercise Sheet 3, Theoretical Quantum and Atom Optics

University of Hamburg, Prof. P. Schmelcher

To be returned on 13/11/2012, in the tutorials

Exercise 5. Coherent States

The coherent states $\{|\alpha\rangle, \alpha \in \mathbb{C}\}$ are an overcomplete non-orthogonal "basis" of the Fock space:

(a) Show that the orthogonality relation $\langle\alpha|\alpha'\rangle = \delta(\alpha - \alpha')$ does not hold for coherent states by explicitly computing the scalar product $\langle\alpha|\alpha'\rangle$.

(b) Prove the following completeness relation of the coherent states:

$$\int \mathrm{d}^2\alpha \, |\alpha\rangle \langle\alpha| = \pi.$$

(c) Combining the above results, show that for a given coherent state $|\beta\rangle$ there are at least two different ways of expanding it in terms of the coherent states $\{|\alpha\rangle, \alpha \in \mathbb{C}\}$.

Hints:
For the calculations expand the coherent state in Fock states $|\alpha\rangle = \sum_n |n\rangle \langle n|\alpha\rangle$.
Use $\alpha = a\mathrm{e}^{\mathrm{i}\phi}$ and $\mathrm{d}^2\alpha = a da d\phi$ before performing the integration and use the definition of the gamma function to solve the integral.

5 Points

Exercise 6. Squeeze operator, squeezed states

Consider the squeeze operator

$$S(\xi) = \exp\left(\frac{1}{2}\xi^* a^2 - \frac{1}{2}\xi(a^\dagger)^2\right),$$

where $\xi = re^{i\theta}$ is a complex number.

(a) Show that $S^\dagger(\xi) = S(-\xi)$. Further conclude that $S(-\xi) = S^{-1}(\xi)$ by showing that $S(\xi)S(-\xi) = 1$.

(b) Now consider the squeezed coherent state $|\alpha, \xi\rangle = S(\xi)D(\alpha)|0\rangle$. Explicitly calculate the expectation values $\langle a\rangle = \langle\alpha, \xi|a|\alpha, \xi\rangle$ and $\langle a^2\rangle = \langle(a^\dagger)^2\rangle^*$.

(c) Y_1 and Y_2 are the rotated quadrature variables, see the lecture. Explicitly calculate the variances $(\triangle Y_1)^2$ and $(\triangle Y_2)^2$ and show that $\triangle Y_1 \triangle Y_2 = \frac{1}{4}$.

(d) There are (at least) two ways for creating a coherent state $|\alpha\rangle$ out of the vacuum $|0\rangle$. One can either employ the unitary displacement operator, i.e. $|\alpha\rangle = \hat{D}(\alpha)|0\rangle$, or one can write $|\alpha\rangle = \tilde{N}\, e^{\alpha\hat{a}^\dagger}|0\rangle$ with $\tilde{N} = 1/||e^{\alpha\hat{a}^\dagger}|0\rangle||$ (which is fun to prove). Similarly, one can generate the squeezed state $|\xi\rangle = \hat{S}(\xi)|0\rangle$ by means of the simpler operator $e^{-\zeta/2\,(\hat{a}^\dagger)^2}$ as follows: The choice $\zeta = \tanh(|\xi|)\frac{\xi}{|\xi|}$ leads to $|\xi\rangle = N\, e^{-\zeta/2\,(\hat{a}^\dagger)^2}|0\rangle$ with $N = 1/||e^{-\zeta/2\,(\hat{a}^\dagger)^2}|0\rangle||$.

Consider the Hamiltonian $\hat{H} = \omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})$ and a squeezed state as the initial condition, i.e. $|\psi(t = 0)\rangle = |\xi\rangle$. Your are now asked to solve the time-dependent Schrödinger equation and to interpret your solution $|\psi(t)\rangle$.

5 Points