Winter term 2012/13

**Exercise Sheet 12, Theoretical Quantum and Atom Optics**

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To be returned on Tuesday, 29/01/2013, in the tutorials

**Exercise 22.** Double-well tunneling

By superimposing a harmonic trap with a narrow blue-detuned laser beam of Gaussian profile, it is possible to create a symmetric double-well potential like (here for a one-dimensional setting):

$$V(x) = \frac{m\omega_0^2}{2}x^2 + b\,e^{-\frac{x^2}{2w^2}}. \tag{1}$$

Setting $\hbar = 1$, let us assume that we have found the ground and first excited state of the corresponding one-body problem $\hat{h}|\psi_\pm\rangle = \epsilon_{0/1}|\psi_\pm\rangle$ with $\hat{h} = -\frac{1}{2m}\partial_x^2 + V(x)$ and $\psi_+(x)$ ($\psi_-(x)$) denoting the symmetric (antisymmetric) ground (first excited) state. It is well known that for high enough barriers, the eigenenergies happen to be pairwise quasi-degenerate up to a certain energy. In particular, we would like to focus on a regime where the level splitting $\epsilon_1 - \epsilon_0 \ll \epsilon_2$. Now let us consider a BEC in this trap at such low temperatures and with such a low strength $U_0$ of the repulsive interaction that essentially no boson can thermally or via the interaction be excited into higher single particle states than the first excited one. Thus we may restrict ourselves to a so called two-mode approximation. Rather than taking the single-particle eigenstates $|\psi_\pm\rangle$ as the two modes for building up the many-body Fock-space, we consider the linear combinations:

$$|\phi_{1/2}\rangle = \frac{1}{\sqrt{2}}\left(|\psi_+\rangle \pm |\psi_-\rangle\right), \tag{2}$$

which are quite localized in the left (right) well for high enough barriers, i.e. $|\phi_1(x)\phi_2(x)| \approx 0$. Please note that these states are exactly orthonormal. Associating the annihilation/creation operators $\hat{a}_i^{(\dagger)}$ with the states $|\phi_i\rangle$, we may write for the bosonic field operators:

$$\hat{\psi}(x) = \phi_1(x)\,\hat{a}_1 + \phi_2(x)\,\hat{a}_2 + ..., \tag{3}$$

where "..." represents energetically frozen out modes.

(a) Use the inverse relation $\hat{a}_i = \int \mathrm{d}x\,\phi_i^*(x)\hat{\psi}(x)$ and the commutation relations for the bosonic field operators $[\hat{\psi}(x), \hat{\psi}(y)^\dagger] = \delta(x-y)$, $[\hat{\psi}(x), \hat{\psi}(y)] = 0$ to show that

$$[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}, \qquad [\hat{a}_i, \hat{a}_j] = 0.$$

(b) For high enough barriers, show that the exact many-body Hamiltonian:

$$\hat{\hat{H}} = \int \mathrm{d}x\,\left(\hat{\psi}^\dagger(x)\,\hat{h}\,\hat{\psi}(x) + \frac{U_0}{2}\,\hat{\psi}^\dagger(x)\hat{\psi}^\dagger(x)\hat{\psi}(x)\hat{\psi}(x)\right) \tag{4}$$

can be approximated by:

$$\hat{\hat{H}} = \frac{\epsilon_0 + \epsilon_1 - U}{2}\hat{N} - J\left(\hat{a}_1^\dagger\hat{a}_2 + \hat{a}_2^\dagger\hat{a}_1\right) + \frac{U}{2}\left(\hat{N}_1^2 + \hat{N}_2^2\right), \tag{5}$$

with $\hat{N}_i = \hat{a}_i^\dagger\hat{a}_i$ being the occupation number operator of the $i$-th well. Determine the parameters $U$ and $J$, interpret them and discuss their signs. Finally, we renormalize the Hamiltonian with respect to the system's energy (for definite particle number) in case of $J = U = 0$:

$$\hat{H} := \hat{\hat{H}}(U,J) - \hat{\hat{H}}(0,0) = -\frac{U}{2}\hat{N} - J\left(\hat{a}_1^\dagger\hat{a}_2 + \hat{a}_2^\dagger\hat{a}_1\right) + \frac{U}{2}\left(\hat{N}_1^2 + \hat{N}_2^2\right), \tag{6}$$

which is the famous two-site *Bose-Hubbard model Hamiltonian* of the lecture.

**Please turn the page.**

(c) Based on the Hamiltonian of Eq. (6), find the equations of motion for the annihilation operators in the Heisenberg picture.

(d) Now we assume that the creation and annihilation operators can be approximated by classical fields and replace

$$\hat{a}_j(t) \to a_j(t) = \sqrt{N_j(t)} e^{i\theta_j(t)}, \qquad j = 1, 2. \tag{7}$$

Find the ensuing classical equations of motion for the population imbalance $n := (N_1 - N_2)/2$ and the relative phase $\theta := \theta_1 - \theta_2$.

(e) Show that the classical approximation can also be made on the level of the Hamiltonian:

Plug the transformation from Eq. (7) into the Hamiltonian (6) and check that taking $(n, \theta)$ to be conjugate variables, the correct equations of motion follow from

$$\dot{n} = \frac{\partial H}{\partial \theta}, \qquad \dot{\theta} = -\frac{\partial H}{\partial n}.$$

(f) Find the fixed points of this dynamical system, i.e. points $(n_0, \theta_0)$ for which $\dot{n}(n_0, \theta_0) = 0$ and $\dot{\theta}(n_0, \theta_0) = 0$.

Does the number of these fixed points depend on the parameters $N$, $J$ and $U$?

**10 Points**