Winter term 2012/13

Exercise Sheet 11, Theoretical Quantum and Atom Optics

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To be returned on Tuesday, 22/01/2013, in the tutorials

Exercise 21. Collective excitations of Bose-Einstein condensates

Consider a repulsively interacting Bose-Einstein condensate ($U_0 > 0$) in an isotropic 3-dimensional harmonic trap: $V(\boldsymbol{r}) = \frac{m\omega_0^2}{2} r^2$. Furthermore, let us assume that the number of atoms is so large that we can apply the Thomas-Fermi approximation leading to the ground state density profile:

$$n_0(\boldsymbol{r}) = \frac{\mu - V(\boldsymbol{r})}{U_0} \quad \text{for } r \leq R = \left(\frac{2\mu}{m\omega_0^2}\right)^{\frac{1}{2}}. \tag{1}$$

In the following, we aim at discussing collective excitations of the atom cloud, for which the hydrodynamic formulation of the Gross-Pitaevskii theory turns out to be a convenient approach (cf. sheet 8). With $\psi = \sqrt{n(\boldsymbol{r},t)}e^{i\varphi(\boldsymbol{r},t)}$, we have obtained:

$$\frac{\partial n}{\partial t} = -\nabla(n\mathbf{v}), \tag{2}$$

$$-\hbar\frac{\partial \varphi}{\partial t} = -\frac{\hbar^2}{2m\sqrt{n}}\Delta\sqrt{n} + \frac{1}{2}m\mathbf{v}^2 + V + nU_0, \tag{3}$$

where the velocity field is given as $\boldsymbol{v} = \frac{\hbar}{m}\nabla\varphi$.

(a) Why can we neglect the quantum pressure term $\propto (\Delta\sqrt{n})/\sqrt{n}$ here?

(b) Show that $\psi_0 = \sqrt{n_0(\boldsymbol{r})}\,\exp(-i\mu t/\hbar)$ solves the Eqs. (2,3) within this approximation.

(c) Consider a small perturbation of this stationary solution, $n(\boldsymbol{r},t) = n_0(\boldsymbol{r}) + \delta n(\boldsymbol{r},t)$, $\varphi(\boldsymbol{r},t) = -\mu t/\hbar + \delta\varphi(\boldsymbol{r},t)$, and linearize Eqs. (2,3) around the stationary solution. Show that the density fluctuation obeys:

$$\partial_t^2 \delta n = \frac{U_0}{m}\left((\nabla n_0)\cdot(\nabla\delta n) + n_0\Delta\delta n\right). \tag{4}$$

(d) Make a separation ansatz $\delta n = f(\boldsymbol{r})\,e^{-i\omega t}$ and show that f satisfies:

$$\omega^2 f = \frac{1}{m}\left((\nabla V)\cdot(\nabla f) - (\mu - V)\Delta f\right). \tag{5}$$

Explicate this equation for the given harmonic trap.

(e) Introduce spherical coordinates $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$. Check that Eq. (5) is solved by the ansatz

$$f = Cr^l Y_{lm}(\theta,\phi), \qquad C \in \mathbb{R}, l \in \mathbb{N}, m \in \{-l, -l+1, \ldots, l\}. \tag{6}$$

where Y_{lm} denote the spherical harmonics, and find the corresponding eigenfrequency ω. Look at this solution in more detail for parameters $l = 1, m = 0$. Show that for an infinitesimally small amplitude of the perturbation, the real part of the time-dependent total density can be written as $\Re\left[n(\boldsymbol{r},t)\right] = n_0\left(\boldsymbol{r} - \epsilon\cos(\omega_0 t)\boldsymbol{e}_z\right)$ with a (small) constant ϵ. What does this oscillation look like?

(f) Check that the ansatz $f = C\left(1 - \frac{5}{3}\frac{r^2}{R^2}\right)$ also solves Eq. (5). What does the corresponding oscillation of the condensate density look like?

10 Points