Winter term 2012/13

Exercise Sheet 10, Theoretical Quantum and Atom Optics

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To be returned on Tuesday, 15/01/2013, in the tutorials

Exercise 20. Vortices in Bose-Einstein Condensates

Let $\psi(\boldsymbol{r})$ denote the Gross-Pitaevskii wavefunction of a repulsively interacting condensate *in free space*. Certainly, $\psi(\boldsymbol{r})$ should be a single-valued quantity as any wavefunction. In particular, the value of $\psi(\boldsymbol{r})$ must coincide with the value of the wavefunction if one starts at $\boldsymbol{r}$ and returns to $\boldsymbol{r}$ after moving along any closed loop ζ.

(a) Consider the Gross-Pitaevskii wavefunction in its hydrodynamic representation, i.e. $\psi(\boldsymbol{r}) = f(\boldsymbol{r})\, e^{i\phi(\boldsymbol{r})}$. Show that for any closed loop ζ the circulation:

$$C_\zeta[\boldsymbol{v}] = \oint_\zeta d\boldsymbol{r}\ \boldsymbol{v}(\boldsymbol{r}) \tag{1}$$

is quantized, where $\boldsymbol{v}(\boldsymbol{r}) = \frac{\hbar}{m}\nabla\phi(\boldsymbol{r})$ denotes the local velocity of the condensate. Quantization of the circulation means that there is a constant α such that $C_\zeta[\boldsymbol{v}] = \alpha\, l$ for some $l \in \mathbb{Z}$. Determine α.

(b) Now show that applying Stokes' theorem to equation (1) *in a naive way* would lead to the conclusion that $l = 0$ always. This, however, contradicts the experimental observations that BECs can feature vortices[1] and even turbulence! So why is Stokes' theorem not applicable in this situation? Please be reminded of the exercise sheet 4 ("the phase operator"). There, one central result was that it is only possible to define an approximately hermitian phase operator if the probability of having a zero density vanishes. So how does the density profile of a vortex have to look like qualitatively?

(c) Now let's turn to a 2-dimensional BEC living in the xy-plane. Suppose that there is a vortex of so-called *charge* l at the origin. Introducing polar coordinates (ρ, φ), assume that the phase field $\phi(\boldsymbol{r})$ is independent of ρ. Make the simplest ansatz for $\phi(\boldsymbol{r})$ leading to a vortex of charge l and determine the corresponding velocity field $\boldsymbol{v}(\boldsymbol{r})$!

(d) Finally, we aim at calculating the energy of a single vortex of charge l, which is defined as the difference of the energy of a uniform BEC with and without vortex. For obtaining a finite energy, we should better consider a disk of radius R out of the uniform condensate, where the vortex shall lie in the origin of this disk. Show that the energy functional for the uniform condensate with the vortex equals:

$$E_l[f] = 2\pi \int_0^R d\rho\,\rho \left[\frac{\hbar^2}{2m}(\partial_\rho f)^2 + \frac{\hbar^2}{2m} l^2 \frac{f^2}{\rho^2} + \frac{U_0}{2} f^4\right], \tag{2}$$

where an isotropic density distribution $f(\rho)$ has been assumed. The density profile can be determined by minimizing $E_l[f]$ under the constraint that $\int d\rho\, \rho f(\rho)$ equals the total particle number. The resulting ODE, however, can only be handled numerically. In contrast to this, the energy of the BEC without vortex, $E_{l=0}[f_0]$, can be easily calculated since f_0 is a constant. One can show that for $R \gg \xi = \sqrt{\hbar^2/2mU_0 f_0^2}$ (ξ is the so called healing length) the energy of the singly charged vortex is given by:

$$\varepsilon_l \equiv E_l[f] - E_{l=0}[f_0] \approx \pi \frac{\hbar^2}{m} f_0^2\, l^2\, \ln\frac{R}{|l|\xi} \tag{3}$$

Would it be energetically favourable to have a single vortex of (not too large) charge $l > 0$ or l spatially well separated singly charged vortices?

10 Points

[1] A BEC is said to have at least one vortex within the boundary ζ if $C_\zeta[\boldsymbol{v}] \neq 0$.