SS 2012

Exercise Sheet 1, Theoretical Quantum and Atom Optics

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To be returned on 16/04/2012, in the tutorials

Exercise 1. Wave equation

The vector potential $\boldsymbol{A}$ in an empty cubic cavity of side L satisfies the wave equation

$$\boldsymbol{\nabla}^2 \boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} = 0, \tag{1}$$

together with the Coulomb gauge condition $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$. Consider a solution of the following form

$$\begin{aligned} A_x(\boldsymbol{r}, t) &= A_x(t)\cos(k_x x)\sin(k_y y)\sin(k_z z) \\ A_y(\boldsymbol{r}, t) &= A_y(t)\sin(k_x x)\cos(k_y y)\sin(k_z z) \\ A_z(\boldsymbol{r}, t) &= A_z(t)\sin(k_x x)\sin(k_y y)\cos(k_z z), \end{aligned}$$

where $k_x = 2\pi n_x/L$, $k_y = 2\pi n_y/L$, $k_z = 2\pi n_z/L$, with $n_x, n_y, n_z \in \mathbb{Z}$.

(a) Determine the condition the functions $A_x(t)$, $A_y(t)$, $A_z(t)$ have to satisfy so that the above is actually a solution of the wave equation. From this condition, determine explicitly the functions $A_x(t)$, $A_y(t)$, $A_z(t)$. (2 points)

(b) Determine the electric and the magnetic field and show that the fields fulfill the boundary conditions for a conducting cavity, i.e. the tangential components of the electric field and the normal components of the magnetic field have to vanish at the boundaries of the cavity. (2 points)

(c) Show that it is not possible that two of the integers n_x, n_y, n_z are zero. (1 point).

5 Points

Exercise 2. Commutator relation

Prove the following relation which was used in the lecture

$$[E_x(\mathbf{r}, t), H_y(\mathbf{r}', t)] = -\mathrm{i}\hbar c^2 \frac{\partial}{\partial z}\delta(\mathbf{r} - \mathbf{r}')$$

Hint: Prove the following relation, using spherical coordinates $\mathbf{k} = k(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ and choosing unit vectors perpendicular to $\mathbf{k}$ and to each other in the form of $\hat{\epsilon}_{\mathbf{k}}^{(1)} = (\sin\phi, -\cos\phi, 0)$ and $\hat{\epsilon}_{\mathbf{k}}^{(2)} = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)$:

$$\sum_{\lambda=1}^{2} \hat{\epsilon}_{\mathbf{k}i}^{\lambda}\hat{\epsilon}_{\mathbf{k}j}^{\lambda} = \delta_{ij} - \frac{k_i k_j}{k^2}$$

and substitute the k-sum by an integral ($\sum_{\mathbf{k}} \to \frac{V}{(2\pi)^3}\int \mathrm{d}^3k$).

5 Points