Winter term 2012/13

**Exercise Sheet 1, Theoretical Quantum and Atom Optics**

University of Hamburg, Prof. P. Schmelcher

To be returned on Tuesday, 30/10/2012, in the tutorials

**Exercise 1.** Wave equation

The vector potential $\boldsymbol{A}$ in an empty cubic cavity of side $L$ satisfies the wave equation

$$\boldsymbol{\nabla}^2 \boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} = 0,$$

together with the Coulomb gauge condition $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$. Consider a solution of the following form

$$\begin{aligned} A_x(\boldsymbol{r}, t) &= A_x(t) \cos(k_x x) \sin(k_y y) \sin(k_z z) \\ A_y(\boldsymbol{r}, t) &= A_y(t) \sin(k_x x) \cos(k_y y) \sin(k_z z) \\ A_z(\boldsymbol{r}, t) &= A_z(t) \sin(k_x x) \sin(k_y y) \cos(k_z z), \end{aligned}$$

where $k_x = 2\pi n_x/L$, $k_y = 2\pi n_y/L$, $k_z = 2\pi n_z/L$, with $n_x, n_y, n_z \in \mathbb{Z}$.

(a) Determine the condition the functions $A_x(t)$, $A_y(t)$, $A_z(t)$ have to satisfy so that the above is actually a solution of the wave equation. From this condition, determine explicitly the functions $A_x(t)$, $A_y(t)$, $A_z(t)$. (2 points)

(b) Determine the electric and the magnetic field and show that the fields fulfill the boundary conditions for a conducting cavity, i.e. the tangential components of the electric field and the normal components of the magnetic field have to vanish at the boundaries of the cavity. (2 points)

(c) What happens if two of the integers $n_x, n_y, n_z$ are zero? (1 point).

**5 Points**

**Exercise 2.** Commutator relation

Prove the following relation which was used in the lecture

$$[E_x(\mathbf{r}, t), H_y(\mathbf{r}', t)] = -\mathrm{i}\hbar c^2 \frac{\partial}{\partial z} \delta(\mathbf{r} - \mathbf{r}').$$

*Hint*: Prove the following relation, using spherical coordinates $\mathbf{k} = k(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ and choosing unit vectors perpendicular to $\mathbf{k}$ and to each other in the form of $\hat{\epsilon}_{\mathbf{k}}^{(1)} = (\sin\phi, -\cos\phi, 0)$ and $\hat{\epsilon}_{\mathbf{k}}^{(2)} = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)$:

$$\sum_{\lambda=1}^{2} \left(\hat{\epsilon}_{\mathbf{k}}^{\lambda}\right)_i \left(\hat{\epsilon}_{\mathbf{k}}^{\lambda}\right)_j = \delta_{ij} - \frac{k_i k_j}{k^2},$$

where $i, j \in \{x, y, z\}$, and substitute the $k$-sum by an integral ($\sum_{\mathbf{k}} \to \frac{V}{(2\pi)^3} \int \mathrm{d}^3 k$).

**5 Points**