8.2 Interferences of two condensates

Demonstrate wave nature of BECs by letting overlap (expand) two initially separated clouds (condensates). Evolution of expanding clouds $\Longleftrightarrow$ Deeper understanding of the conditions under which interferences occur.

Different setups here

(1) Phases of different clouds are locked

(2) Two clouds with fixed particle numbers: no locked phases

8.2.1 Phase-locked sources

Assume two non-overlapping sources $\psi_1(\bar{r},t)$, $\psi_2(\bar{r},t)$.

Coherence between the clouds $\Rightarrow$ State can be described by a single condensate wave function ψ

$$\psi(\bar{r},t) = \underset{\uparrow}{\sqrt{N_1}}\, \psi_1(\bar{r},t) + \underset{\uparrow}{\sqrt{N_2}}\, \psi_2(\bar{r},t)$$

expectation values of # of particles in the two clouds

Expansion: Overlapping and interfering clouds

Neglect interaction effects in the overlap region

$$\Rightarrow n(\bar{r},t) = |\psi(\bar{r},t)|^2 = \left|\sqrt{N_1}\,\psi_1(\bar{r},t) + \sqrt{N_2}\,\psi_2(\bar{r},t)\right|^2$$

$$= N_1\,|\psi_1(\bar{r},t)|^2 + N_2\,|\psi_2(\bar{r},t)|^2 + 2\sqrt{N_1 N_2}\,\underset{\Uparrow}{\mathrm{Re}}\left(\psi_1(\bar{r},t)\,\psi_2^*(\bar{r},t)\right)$$

Interference is caused by the spatial dependence of the phases of the wave functions for the individual clouds

Example: Take Gaussian wave packets of width R_0 centered on the points $\bar{r} = \pm \bar{d}/2$

Neglecting interactions & external potentials yields

$$\psi_1 = (\pi R_t^2)^{-3/4} \, e^{i(\phi_1 + \delta_t)} \, \exp\left[- \frac{(\bar{r} - \bar{d}/2)^2 (1 - i\hbar t / m R_0^2)}{2 R_t^2} \right]$$

and

$$\psi_2 = (\pi R_t^2)^{-3/4} \, e^{i(\phi_2 + \delta_t)} \, \exp\left[- \frac{(\bar{r} + \bar{d}/2)^2 (1 - i\hbar t / m R_0^2)}{2 R_t^2} \right]$$

m ... particle mass

ϕ_1, ϕ_2 initial phases of condensates

$\tan \delta_t = -\hbar t / m R_0^2$

and $R_t^2 = R_0^2 + \left(\frac{\hbar t}{m R_0} \right)^2$ (free particle SE)

From this we conclude that the interference term reads as follows

$$\mathrm{Re}\left[\psi_1(\bar{r},t) \, \psi_2^*(\bar{r},t) \right] \propto e^{-r^2/R_t^2} \cos\left(\underbrace{\frac{\hbar}{m} \frac{\bar{r}\bar{d}}{R_0^2 R_t^2} t}_{\text{can oscillate rapidly}} + \phi_1 - \phi_2 \right)$$

Planes of constant phase: $(\bar{r}\bar{d}) = 0$; $\bar{r} \perp \bar{d}$

Positions of maxima depend on relative phase of the two condensates.

Distance between maxima: Assume $\vec{d} \parallel \vec{e}_z$

drops due to difference taken

$$\frac{\hbar}{m}\frac{(\overset{\Delta z}{\Uparrow}{\vec{r}})\,d\,t}{R_0^2 R_t^2} + \cancel{\phi_1 - \phi_2} = 2\pi$$

$$\rightsquigarrow \quad \Delta z = 2\pi \frac{m R_t^2 R_0^2}{\hbar t d}$$

After a significant time of expansion (size is multiple compared to the original one) $R_t \simeq \frac{\hbar t}{m R_0}$ and therefore

$$\Delta z \simeq \frac{2\pi \hbar t}{m d}$$

These interference patterns are indeed observed in the experiment!

However: Interference patterns occur also in case of completely decoupled condensates before they expand and overlap.

<u>Thought experiment</u>

Two condensates are trapped and initially isolated from each other: DW with a very high barrier.

- No interaction
- $N/2$ particles in each well.

<u>Aim</u>: Calculate probability of finding a certain # of atoms in the states

$$\psi_\pm = (\underset{\substack{\uparrow}}{\psi_1} \pm \underset{\substack{\uparrow}}{\psi_2})/\sqrt{2}$$

individual well states!

Initially: Fock state $|N_1, N_2\rangle$ corresponding to a definite # of particles in each well!

$N = N_1 + N_2 \gg n$ (# of detected particles)

Probability of detecting first atom in state $|\psi_+\rangle$

$$|\psi_{1+}\rangle = \mathcal{N}\hat{a}_+ |N_1, N_2\rangle = \frac{\mathcal{N}}{\sqrt{2}}(\hat{a}_1 + \hat{a}_2)|N_1, N_2\rangle$$

$$= \frac{\mathcal{N}}{\sqrt{2}}\left(\sqrt{N_1}\,|N_1-1, N_2\rangle + \sqrt{N_2}\,|N_1, N_2-1\rangle\right)$$

$$|\psi_{1-}\rangle = \mathcal{N}\hat{a}_- |N_1, N_2\rangle = \mathcal{N}/\sqrt{2}\left(\sqrt{N_1}\,|N_1-1, N_2\rangle - \sqrt{N_2}\,|N_1, N_2-1\rangle\right)$$

$$\Rightarrow \langle\psi_{1+}|\psi_{1+}\rangle = \langle\psi_{1-}|\psi_{1-}\rangle$$

$\Rightarrow$ Probability of detecting first atom in state $|\psi_+\rangle$ is $\frac{1}{2}$.

However: Probability of detecting n atoms consecutively in the $|\psi_+\rangle$ state is <u>not</u> $(1/2)^n$ since

$$|\psi_{2+}\rangle = \hat{a}_+|\psi_{1+}\rangle = \frac{1}{(2N)^{1/2}}\Big[N_1^{1/2}(N_1-1)^{1/2}\,|N_1-2, N_2\rangle$$

$$+ 2(N_1N_2)^{1/2}|N_1-1, N_2-1\rangle + N_2^{1/2}(N_2-1)^{1/2}|N_1, N_2-2\rangle\Big]$$

but

$$|\psi_{2+-}\rangle = \hat{a}_-|\psi_{1+}\rangle = \frac{1}{(2N)^{1/2}}\Big[N_1^{1/2}(N_1-1)^{1/2}\,|N_1-2, N_2\rangle$$

$$- N_2^{1/2}(N_2-1)^{1/2}\,|N_1, N_2-2\rangle\Big]$$

$\uparrow$

cross-term missing: probability not $\frac{1}{2}$

$\Rightarrow$ For $(N_1 = N_2 = \frac{N}{2})$ case: $P_2^{++} = \frac{3}{4}$, $P_2^{+-} = \frac{1}{4}$

Extending this yields for the probability to measure n-atoms subsequently in the $|\psi_+\rangle$ state

$$P_n^{+\ldots+} = \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2n-1}{2n} = \frac{(2n)!}{(2^n n!)^2} \approx 1/(\pi n)^{1/2}$$

Stirling formula, large n
$\ln(n!) = n\ln n - n + \ln(2\pi n)$

$\Rightarrow$ Conclusion:

Even when there is initially no phase relationship/correlation between the phases of the two clouds, after measurement of n atoms in the $|\psi_+\rangle$ state the resulting states 'approaches' the pure + state, i.e.

$$[(N-n)!]^{-1/2} (\hat{a}_+^\dagger)^{(N-n)} |0\rangle \quad \text{as } n \text{ increases.}$$

Comment (important!)

Note that the same probability would have been obtained if the condensate would have been described by a phase coherent 'two cloud' state

$$\psi(\vec{r},t) = (\psi_1 e^{i\phi_1} + \psi_2 e^{i\phi_2})/\sqrt{2}.$$

$\Rightarrow$ Number state and semiclassical description provide the same result!

Process of detection builds up a phase coherence!

Phase states

Let us introduce a general superposition state

$$\Psi_\phi(r) = \frac{1}{\sqrt{2}}\left[\Psi_1(\bar{r})\, e^{i\phi/2} + \Psi_2(\bar{r})\, e^{-i\phi/2}\right]$$

Overall phase is put to zero.

N particles in the above state reads

$$|\phi, N\rangle = (2^N N!)^{-1/2}\, (\hat{a}_1^+ e^{i\phi/2} + \hat{a}_2^+ e^{-i\phi/2})^N |0\rangle$$

where $\hat{a}_1^+/\hat{a}_2^+$ create particles in cloud 1/2, respectively.

$$\hat{a}_i^+ = \int d\bar{r}\, \Psi_i(\bar{r})\, \hat{\psi}^+(\bar{r})$$

No interactions among clouds: time evolution is calculated in each cloud separately

$$|\phi, N, t\rangle = \frac{1}{\sqrt{N!}}\left[\int d\bar{r}\, \Psi_\phi(\bar{r},t)\, \hat{\psi}^+(\bar{r})\right]^N |0\rangle$$

Phase states form an overcomplete set.

Overlap

$$\langle \phi', N=1 | \phi, N=1\rangle = \int d\bar{r}\, \Psi_{\phi'}^*(\bar{r},t)\, \Psi_\phi(\bar{r},t)$$

where

$$\Psi_{\phi'}^*(\bar{r},t)\Psi_\phi(\bar{r},t) = \frac{1}{2}|\Psi_1(\bar{r},t)|^2 e^{i(\phi-\phi')/2} + \frac{1}{2}|\Psi_2(\bar{r},t)|^2 \cdot e^{-i(\phi-\phi')/2} + \mathrm{Re}\left[\Psi_1(\bar{r},t)\Psi_2^*(\bar{r},t)\, e^{i(\phi+\phi')/2}\right]$$

normalized

$$= \frac{1}{2}\left[|\Psi_1(\bar{r},t)|^2 + |\Psi_2(\bar{r},t)|^2\right]\cos((\phi-\phi')/2)$$

$$+\frac{i}{2}\left[|\Psi_1(\bar{r},t)|^2 - |\Psi_2(\bar{r},t)|^2\right]\sin((\phi-\phi')/2) + \mathrm{Re}\left[\Psi_1\Psi_2^*\, e^{i(\phi+\phi')/2}\right]$$

~ Zero due to rapid spatial oscillations

$$\Rightarrow \quad \langle \phi', N=1 | \phi, N=1 \rangle = \cos((\phi-\phi')/2)$$

Maximum for $\phi = \phi'$, but also nonzero for $\phi \neq \phi$!

Overlap of n-particle phase states

$$\langle \phi', N | \phi, N \rangle = \cos^N((\phi - \phi')/2)$$

More Rapid fall off with increasing N!

(ϕ, ϕ') - localized overlap; otherwise approx. orthogonal states.

Lets evaluate the density operator in a phase state

$$\hat{\psi}(\bar{r}) | \phi, N, t \rangle = \sqrt{N} \, \psi_\phi(\bar{r}, t) \, | \phi, N-1, t \rangle$$

since $\hat{\psi}(\bar{r}) = \sum_i \psi_i a_i$ where the representation containing ψ_ϕ is chosen

and

$$n(\bar{r}, t) = \langle \phi, N, t | \hat{\psi}^+(\bar{r}) \hat{\psi}(\bar{r}) | \phi, N, t \rangle = \frac{N}{2} | \psi_1 e^{i\phi/2} + \psi_2 e^{-i\phi/2} |^2$$

which contains an interference term!

8.2.2 Clouds with definite particle numbers

Let us consider the Fock state

$$| N_1, N_2 \rangle = \frac{1}{\sqrt{N_1! \, N_2!}} \left(\hat{a}_1^+\right)^{N_1} \left(\hat{a}_2^+\right)^{N_2} | 0 \rangle$$

First calculate the expectation value of the density operator $\hat{\psi}^+(x) \hat{\psi}(x)$ by noticing

$$\hat{\psi}(r) | N_1, N_2, t \rangle = \sqrt{N_1} \, \psi_1(\bar{r}, t) | N_1 - 1, N_2, t \rangle + \sqrt{N_2} \, \psi_2(\bar{r}, t) | N_1, N_2 - 1, t \rangle$$

$$\Rightarrow \quad n(\bar{r}) = \langle N_1, N_2, t | \hat{\psi}^+(\bar{r}) \hat{\psi}(\bar{r}) | N_1, N_2, t \rangle$$

$$= N_1 |\psi_1|^2 + N_2 |\psi_2|^2$$

i.e. without interference terms.

However, this does not mean that there are no interference effects for Fock states.

<u>Note</u>

Exp. → Prepare clouds, expand, detect positions of atoms:
'One-shot' experiment has to be repeated many times to describe quantum expectation values.

Above absence of interference pattern holds only for many shot average!

Indeed, the single shot measurement corresponds more to a higher order correlation function.

<u>Example</u>: Two particle correlation function

$\hat{=}$ Probability amplitude of destroying particles at $\bar{r}_*$ and $\bar{r}'$ and then creating them again at the same positions

$$\langle N_1, N_2, t | \hat{\psi}^+(\bar{r}) \psi^+(\bar{r}') \psi(\bar{r}) \psi(\bar{r}') | N_1, N_2, t \rangle$$

$$= \left[N_1 |\psi_1(\bar{r},t)|^2 + N_2 |\psi_2(\bar{r},t)|^2 \right] \left[N_1 |\psi_1(\bar{r}',t)|^2 + N_2 |\psi_2(\bar{r}',t)|^2 \right]$$

$$- N_1 |\psi_1(\bar{r},t)|^2 |\psi_1(\bar{r}',t)|^2 - N_2 |\psi_2(\bar{r},t)|^2 |\psi_2(\bar{r}',t)|^2$$

$$+ 2 N_1 N_2 \, \mathrm{Re} \left[\psi_1^*(\bar{r}',t) \psi_1(\bar{r},t) \psi_2^*(\bar{r},t) \psi_2(\bar{r}',t) \right]$$

:

Correlation is in the last term: Hanbury, Brown & Twiss interferometer

$\Rightarrow$ Coherence is not prerequisite for interference!

<u>Note</u>: (a) the two-particle correlation function for a Fock state would be $\propto 1 + \frac{1}{2}\cos(\Delta\phi_1 - \Delta\phi_2)$

where $\Delta\phi_i = \phi_i(\vec{r},t) - \phi_i(\vec{r}',t)$
$\uparrow$
Phase of $\psi_i(\vec{r},t)$

(b) For two sources with definite phases, the correlation function would be proportional to

$1 + \cos(\Delta\phi_1 - \Delta\phi_2)$

$\Rightarrow$ Only a reduction in phase interference!

<u>In the following</u>: Relate results of a phase state to those of a Fock state.

Use binomial expansion

phases of localized states

$$|\phi, N\rangle = (2^N N!)^{-1/2} \sum_{N_1=0}^{N} e^{+i(2N_1-N)\phi/2} \frac{N!}{(N-N_1)!(N_1)!} \quad (*)$$

counting of combinations

$$\cdot \sqrt{N_1!(N-N_1)!}\, |N_1, N-N_1\rangle$$

$\Uparrow$
comes from acting with the creation operator on $|0\rangle$

<u>Next step</u>: Expand Fock state $|N_1, N_2\rangle = |\frac{N}{2}, \frac{N}{2}\rangle$ in terms of phase states.

Integrating ($*$) over ϕ from 0 to 2π: The only surviving term is $\propto |N/2, N/2\rangle$

$$\Rightarrow \int_0^{2\pi} |\phi, N\rangle \, d\phi \cdot \frac{1}{2\pi} = \underbrace{\left(2^N \tfrac{N}{2}\right)^{-1/2} (N!)^{+1/2}}_{1/(\pi N/2)^{1/4} \text{ for large } N} \left|\tfrac{N}{2}, \tfrac{N}{2}\right\rangle$$

$$\Rightarrow \left|\tfrac{N}{2}, \tfrac{N}{2}\right\rangle = \left(\pi \tfrac{N}{2}\right)^{1/4} \frac{1}{2\pi} \int_0^{2\pi} |\phi, N\rangle \, d\phi \qquad (**)$$

Fock state is a superposition of equally weighted phase states.

<u>Return to thought experiment:</u>

- Initially Fock state $|\tfrac{N}{2}, \tfrac{N}{2}\rangle$
- Series of detections in both the + and − channels

$$n = n_+ + n_-$$

- $\hat{a}_+$ acting on $|\phi, N\rangle \rightarrow \cos(\phi/2)$ factor

  $\hat{a}_-$ " " " $\rightarrow \sin(\phi/2)$ factor

$$\Rightarrow (\hat{a}_+)^{n_+} (\hat{a}_-)^{n_-} |\phi, N\rangle \propto (\cos(\phi/2))^{n_+} (\sin(\phi/2))^{n_-} |\phi, N-n\rangle$$

$(\cos(\phi/2))^{2n_+} (\sin(\phi/2))^{2n_-}$ determines the probability distribution of phase states after $n = n_+ + n_-$ detections.

<u>Maxima:</u> $\frac{\partial}{\partial \phi} \left(\cos^{2n_+}(\phi/2) \sin^{2n_-}(\phi/2)\right) \stackrel{!}{=} 0$

$$-(2n_+)\ \cancel{\cos^{2n_+-1}(\phi/2)}\ \sin^{\cancel{2n_-}\,1}(\phi/2)\ \sin(\phi/2)\cdot\cancel{\tfrac{1}{2}}$$

$$+\ \cos^{\cancel{2n_+}\,1}(\phi/2)\ \cancel{\sin^{2n_--1}(\phi/2)}\cdot(2n_-)\cos(\phi/2)\ \cancel{\tfrac{1}{2}}\ =0$$

$$-(\cancel{2}n_+)\ \sin^2(\phi/2)+\cos^2(\phi/2)\cdot\cancel{2}n_-\ \overset{!}{=}0$$

$$\frac{\sin^2(\phi_0/2)}{\cos^2(\phi_0/2)}=\boxed{\tan^2(\phi_0/2)=\frac{n_-}{n_+}}$$

or equivalently $\cos(\phi_0)=\left(\dfrac{n_+-n_-}{n_++n_-}\right)$

$\Rightarrow$ After a series of measurements, the state of the system will therefore be no longer one where all phases states are equally weighted but phases will be centred around some values ($\pm\,\phi_0$) with above conditions.

Spread $\propto \dfrac{1}{\sqrt{n}}$.