SS 2012

8. Exercise Sheet, Lecture Theoretical Quantum and Atom Optics

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To be returned on 11/06/2012, in the tutorials

Exercise 15. Lasing without inversion

If you consider a typical laser, usually population inversion is required to achieve lasing, i.e. the population of the excited states has to be larger than the one of the lowest state.

(a) Give a short (!) argument why a typical laser requires population inversion.

Now, in this exercise we will deal with the concept of "lasing without inversion". To this end, consider again the three level system in the Λ configuration you know from the lectures (Two low lying states $|1\rangle$ and $|3\rangle$ and one excited state $|2\rangle$. The transition from $|1\rangle$ to $|3\rangle$ and vice versa is dipole forbidden. Ω_{12}, ϕ_{12} and Ω_{23}, ϕ_{23} are the absolute value and the complex phase of the Rabi frequencies between $|1\rangle$ and $|3\rangle$ and between $|2\rangle$ and $|3\rangle$, respectively). You have seen that the time evolution of the probability amplitudes $c_i(t)$ for finding the atom in level i is governed by the following system of differential equations (resonant case, zero detunings):

$$\begin{aligned}
\dot{c}_1(t) &= \frac{i}{2}\Omega_{12}e^{i\phi_{12}}c_2(t) \\
\dot{c}_2(t) &= \frac{i}{2}\left(\Omega_{12}e^{-i\phi_{12}}c_1(t) + \Omega_{23}e^{i\phi_{23}}c_3(t)\right) \\
\dot{c}_3(t) &= \frac{i}{2}\Omega_{23}e^{-i\phi_{23}}c_2(t)
\end{aligned}$$

(b) For the case that initially the sytem is in a superposition of the two lower levels, the solution of this ODE system has been given in the lecture. You are asked to explicitly validate this result here. To do so, show that

$$\begin{aligned}
c_1(t) &= \frac{1}{\Omega^2}\left([\Omega_{23}^2 + \Omega_{12}^2\cos(\Omega t/2)]\cos(\Theta/2) - 2\Omega_{12}\Omega_{23}e^{i(\phi_{12}+\phi_{23}-\psi)}\sin^2(\Omega t/4)\sin(\Theta/2)\right) \\
c_2(t) &= \frac{i}{\Omega}\sin(\Omega t/2)\left[\Omega_{12}e^{-i\phi_{12}}\cos(\Theta/2) + \Omega_{23}e^{-i(\psi-\phi_{23})}\sin(\Theta/2)\right] \\
c_3(t) &= \frac{1}{\Omega^2}\left([\Omega_{12}^2 + \Omega_{23}^2\cos(\Omega t/2)]e^{-i\psi}\sin(\Theta/2) - 2\Omega_{12}\Omega_{23}e^{-i(\phi_{12}+\phi_{23})}\sin^2(\Omega t/4)\cos(\Theta/2)\right)
\end{aligned}$$

solves the above system, subject to the initial condition

$$|\psi(0)\rangle = \sum_{i=1}^{3} c_i(0)|i\rangle = \cos(\Theta/2)|1\rangle + \sin(\Theta/2)e^{-i\psi}|3\rangle.$$

Here, $\Omega^2 = \Omega_{12}^2 + \Omega_{23}^2$ has been introduced.

(c) Now focus on the case where initially the two lower levels $|1\rangle$ and $|3\rangle$ are populated equally, i.e. $\Theta = \pi/2$. We are interested in the probability of finding this system in the upper level $|2\rangle$ after a short time interval has elapsed. To find this probability, assume $\Omega_{12} = \Omega_{23} = \Omega_R$. Show that after the small time step the system has probability zero to be found in the upper level if

$$\phi_{12} + \phi_{23} - \psi = \pm\pi$$

holds. You have recovered the conditions for coherent population trapping stated in the lectures.

(d) Let us turn to the case where the three level system starts in the upper state, i.e. $c_1(0) = c_3(0) = 0$, $c_2(0) = 1$. Use the harmonic ansatz $c_i(t) = a_i\cos(\omega t) + b_i\sin(\omega t)$, where a_i, b_i, ω are constants to be determined, to find the time dependent probability amplitudes for this initial condition.

What is the probability for finding the system still in $|2\rangle$ at time $t \ll 1/\Omega$? Perform a Taylor expansion to second order in Ωt. In a familiar two level system, how would you call the process that makes this probability less than 1?

Now if such a Λ system is put inside a cavity and the atoms are pumped at a certain rate in a coherent superposition of states

$$\rho(t) = \rho_{11} |1\rangle \langle 1| + \rho_{22} |2\rangle \langle 2| + \rho_{33} |3\rangle \langle 3| + \rho_{13} |1\rangle \langle 3| + \rho_{31} |3\rangle \langle 1|, \tag{1}$$

where ρ_{ii}, $i = 1, 2, 3$ are the level populations and $\rho_{ii'}$, $i \neq i'$ are the atomic coherences, we can look at the absorption and emission probabilities.
The probability of emission is given by

$$P_{emission} = P_1 + P_3 = (|\kappa_{2\to 1}|^2 \mathcal{E}^2 + |\kappa_{2\to 3}|^2 \mathcal{E}^2)\rho_{22}, \tag{2}$$

where the κ's are constants depending on the matrix element between the relevant levels and the coupling of the atom with the field (with amplitude $\mathcal{E}$). The absorption probability is given by

$$P_{absorption} = \kappa |c_1 + c_3|^2 \mathcal{E}^2 = \kappa[\rho_{11} + \rho_{33} + \rho_{13} + \rho_{31}]\mathcal{E}^2. \tag{3}$$

Thus, the rate of growth of the laser field amplitude, under appropriate conditions, becomes

$$\dot{\mathcal{E}} = \frac{a}{2}[\rho_{22} - \rho_{11} - \rho_{33} - \rho_{13} - \rho_{31}]\mathcal{E} \tag{4}$$

with some constant a. If the terms ρ_{13} and ρ_{31} cancel ρ_{11} and ρ_{33}, we get

$$\dot{\mathcal{E}} = \frac{a}{2}\rho_{22}\mathcal{E}, \tag{5}$$

which means we can have lasing if only a small fraction ($\rho_{22} > 0$) of atoms is in the excited state. This is called lasing without inversion.

10 Points