Winter term 2012/13

Exercise Sheet 6, Theoretical Quantum and Atom Optics

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To be returned on Tuesday, 04/12/2012, in the tutorials

Exercise 12. Landau-Zener formula

(a) Consider a two-level system described by the time-dependent Hamiltonian

$$H(t) = \begin{pmatrix} -\epsilon & -f \\ -f & \epsilon \end{pmatrix}, \tag{1}$$

with a linearly time dependent sweep $\epsilon = -\alpha t$, where $f, \alpha > 0$ are real constants.

Show that the Schrödinger equation (setting $\hbar \equiv 1$)

$$i\frac{\partial}{\partial t}\begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} = H(t)\begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} \tag{2}$$

leads to the ordinary differential equation

$$\frac{d^2}{dt^2}c_2(t) + \left[f^2 - i\alpha + (\alpha t)^2\right] c_2(t) = 0 \tag{3}$$

for the amplitude $c_2(t)$.

(b) Show that the variable substitution $t \to z(t) = e^{-i\frac{\pi}{4}}(2\alpha)^{\frac{1}{2}}t$ transforms Eq. (3) to the so called Weber equation,

$$\frac{d^2}{dz^2}\tilde{c}_2(z) + \left[\nu + \frac{1}{2} - \frac{1}{4}z^2\right]\tilde{c}_2(z) = 0 \tag{4}$$

with $\nu = \frac{if^2}{2\alpha}$, where $\tilde{c}_2(z) = c_2(z(t))$.

(c) Eq. (4) is solved by the four parabolic cylinder (Weber) functions $D_\nu(z)$, $D_\nu(-z)$, $D_{-\nu-1}(iz)$ and $D_{-\nu-1}(-iz)$, the former two of which are linearly independent for $\nu \notin \mathbb{Z}$.

Verify that $D_\mu(\zeta)$ is a solution of the Weber equation (4) with $\nu = \mu$ if it obeys the following recursive relations for arbitrary ζ and μ:

$$D_{\mu+1}(\zeta) - \zeta D_\mu(\zeta) + \mu D_{\mu-1}(\zeta) = 0\,, \tag{5}$$

$$\frac{d}{d\zeta}D_\mu(\zeta) + \frac{1}{2}\zeta D_\mu(\zeta) - \mu D_{\mu-1}(\zeta) = 0 \tag{6}$$

(d) Assume that, initially, the system is in the state $|1\rangle$, so that the following initial conditions hold:

$$|c_1(t \to -\infty)|^2 = 1, \tag{7}$$

$$|c_2(t \to -\infty)|^2 = 0. \tag{8}$$

During the linear sweep over the avoided crossing, the coupling f causes population transfer from $|1\rangle$ to $|2\rangle$, and the aim is to find the final distribution

$$P_{LZ} \equiv |c_1(t \to \infty)|^2 = 1 - |c_2(t \to \infty)|^2. \tag{9}$$

Among the four solutions of Eq. (4) only the Weber function $D_{-\nu-1}(-iz(t))$ vanishes for $t \to -\infty$. The amplitude $c_2(t)$ thereby fulfills the initial condition (8) if it is written as

$$c_2(t) = \tilde{c}_2(z(t)) = AD_{-\nu-1}(-iz(t)). \tag{10}$$

where A is a normalization constant. Defining $R \equiv \sqrt{2\alpha}t$, the asymptotic expressions for $D_{-\nu-1}(-iz(t))$ in the two limits $t \to \mp\infty$ are given by

$$D_{-\nu-1}(-iz(t \to -\infty)) = e^{-\frac{1}{4}\pi(\nu+1)i} e^{-i\frac{R^2}{4}} R^{-\nu-1} \tag{11}$$

$$D_{-\nu-1}(-iz(t \to +\infty)) = \frac{\sqrt{2\pi}}{\Gamma(\nu+1)} e^{\frac{1}{4}\pi\nu i} e^{i\frac{R^2}{4}} R^{\nu} \tag{12}$$

(i) Find the asymptotic expression $c_1(t \to -\infty)$, by substituting $c_2(t \to -\infty)$ into the Schrödinger equation (2).
Hint: Use the chain rule $\frac{dc_2}{dt} = \frac{dc_2}{dR}\frac{dR}{dt}$.

(ii) Show that $A = \sqrt{\gamma}e^{-\frac{\pi\gamma}{4}}$ fulfills the normalization condition (7), where $\gamma = -i\nu = \frac{f^2}{2\alpha}$.

(iii) Derive the Landau-Zener formula:

$$P_{LZ} = e^{-2\pi\gamma}. \tag{13}$$

Hint: Calculate $|c_2(t \to \infty)|^2$ using the properties of the Gamma function:

$$\Gamma(\pm i\gamma + 1) = \pm i\gamma\Gamma(\pm i\gamma) \tag{14}$$

$$|\Gamma(\pm i\gamma)| = \sqrt{\frac{\pi}{\gamma \sinh \pi\gamma}} \tag{15}$$

10 Points