SS 2012

Excercise Sheet 5, Lecture Theoretical Quantum and Atom Optics

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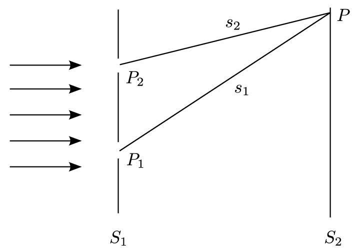
To be returned on 14/05/2012, in the tutorials

Exercise 9. Young's double-slit experiment



Consider the positive frequency part of the field operator at a point P on the screen S_2 at time t as a linear superposition of the field operators present at P_1 and P_2 (position vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, respectively) of screen S_1 at earlier times:

$$E^{(+)}(\boldsymbol{r},t) = K_1 E^{(+)}(\boldsymbol{r}_1, t-t_1) + K_2 E^{(+)}(\boldsymbol{r}_2, t-t_2),$$

where $t_i = s_i/c$ is the traveling time from each pinhole (see figure) and K_1, K_2 are imaginary coefficients that depend on the pinhole size and geometry.

Show that, for statistically stationary fields, i.e. only the time dependence on $\tau = t_1 - t_2$ survives on average, the intensity $\langle I(\boldsymbol{r},t)\rangle = \mathrm{Tr}[\rho E^{(-)}(\boldsymbol{r},t)E^{(+)}(\boldsymbol{r},t)]$ measured at P is given by

$$\langle I(\boldsymbol{r},\tau)\rangle = \langle I^{(1)}(\boldsymbol{r})\rangle + \langle I^{(2)}(\boldsymbol{r})\rangle + 2[\langle I^{(1)}(\boldsymbol{r})\rangle\langle I^{(2)}(\boldsymbol{r})\rangle]^{1/2}\mathrm{Re}[g^{(1)}(\boldsymbol{r}_1,\boldsymbol{r}_2;\tau)],$$

where $\langle I^{(i)}(\boldsymbol{r})\rangle = |K_i|^2 G^{(1)}(\boldsymbol{r}_i,\boldsymbol{r}_i;0)$ and $g^{(1)}$ is the normalized first-order correlation function $G^{(1)}$.

3 Points

Exercise 10. Visibility

Show that the visibility of the interference fringes in Young's experiment (Ex. 9) is given by:

$$U = \frac{2[\langle I^{(1)}(\boldsymbol{r})\rangle\langle I^{(2)}(\boldsymbol{r})\rangle]^{1/2}}{\langle I^{(1)}(\boldsymbol{r})\rangle + \langle I^{(2)}(\boldsymbol{r})\rangle}|g^{(1)}(\boldsymbol{r}_1,\boldsymbol{r}_2;\tau)|.$$

Hint: Use $g^{(1)}$ in its polar representation $g^{(1)}(\boldsymbol{r}_1,\boldsymbol{r}_2;\tau) = |g^{(1)}(\boldsymbol{r}_1,\boldsymbol{r}_2;\tau)|e^{i\alpha(\boldsymbol{r}_1,\boldsymbol{r}_2;\tau)-i\nu_0\tau}$, where the phase factor α varies slowly in space and ν_0 is the field frequency.

3 Points

Exercise 11. Atom-field interaction Hamiltonian

Consider the minimal-coupling Hamiltonian for an electron in an external electromagnetic field,

$$H = \frac{1}{2m}[\boldsymbol{p} - e\boldsymbol{A}(\boldsymbol{r},t)]^2 + eU(\boldsymbol{r},t) + V(r),$$

where the potentials $U, \boldsymbol{A}$ produce the external field, while V binds the electron to a nucleus at $\boldsymbol{r}_0$. Show that, in the radiation gauge: $U(\boldsymbol{r},t) = 0, \nabla\cdot\boldsymbol{A}(\boldsymbol{r},t) = 0$, and in the dipole approximation: $\boldsymbol{k}\cdot\boldsymbol{r} \ll 1$, so that $\boldsymbol{A}(\boldsymbol{r}_0+\boldsymbol{r},t) \approx \boldsymbol{A}(t)\exp(i\boldsymbol{k}\cdot\boldsymbol{r}_0) \Longrightarrow \boldsymbol{A}(\boldsymbol{r},t) \equiv \boldsymbol{A}(\boldsymbol{r}_0,t)$, H can be written in the following forms, where $H_0 = \frac{p^2}{2m} + V(r)$ is the Hamiltonian in the absence of the external field:

(a) $H_r = H_0 - e\boldsymbol{r}\cdot\boldsymbol{E}(\boldsymbol{r}_0,t)$

Hint: Gauge-transform the electronic wave function ψ into $\phi(\boldsymbol{r},t) = \exp\left[-\frac{ie}{\hbar}\boldsymbol{A}(\boldsymbol{r}_0,t)\cdot\boldsymbol{r}\right]\psi(\boldsymbol{r},t)$.

(b) $H_p = H_0 - \frac{e}{m}\boldsymbol{p}\cdot\boldsymbol{A}(\boldsymbol{r}_0,t)$

Hint: Use the fact that the commutator $[\boldsymbol{p},\boldsymbol{A}]$ vanishes in the radiation gauge, and neglect quadratic terms $\sim A^2$.

4 Points