Winter term 2012/13

Exercise Sheet 4, Theoretical Quantum and Atom Optics

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To be returned on Tuesday, 20/11/2012, in the tutorials

Exercise 7. Attempts at constructing a phase operator

You know that a complex number z can be written in polar form as $z = \lambda e^{i\phi}$, where $\lambda = \sqrt{z^* z}$ denotes the modulus of z, and the phase ϕ is a real number from an interval of length 2π.
In many respects, Hermitian operators can be thought of as analogues of real numbers: Take for example the defining identity for a Hermitian operator, $\hat{A} = \hat{A}^\dagger$, which is similar to $z = z^*$ for a real number z. In quantum mechanics you have also seen that Hermitian operators have only real eigenvalues, underlining this connection.
Pushing this analogy a bit further, it seems natural to ask if for any (non-Hermitian) operator $\hat{z}$ in Hilbert space there exists a polar representation $\hat{z} = \hat{\lambda} e^{i\hat{\phi}}$ with Hermitian operators $\hat{\lambda}$, $\hat{\phi}$. In the context of quantum optics, this question is particularly relevant for the creation and annihilation operators $\hat{a}^\dagger$, $\hat{a}$. Dirac made a first attempt at defining a suitable phase operator $\hat{\phi}$ in 1927, and since then there has been a lot of discussion on this - even today the issue of the existence of such a Hermitian quantum phase operator is not fully settled. This exercise will move along the lines of Dirac's original work.

(a) Suppose that a polar decomposition of the bosonic annihilation operator $\hat{a}$ exists, i.e. that one can write

$$\hat{a} = \hat{\lambda} e^{i\hat{\phi}} \tag{1}$$

with Hermitian operators $\hat{\lambda}$, $\hat{\phi}$. Taking the Hermitian conjugate of eq. (1) and making use of the commutator $\left[\hat{a}, \hat{a}^\dagger\right] = 1$, show that $\hat{\lambda}^2 = \hat{n} + 1$, where $\hat{n} = \hat{a}^\dagger \hat{a}$ is the particle number operator. Thus, the polar decompositions read

$$\hat{a} = \sqrt{\hat{n}+1}\, e^{i\hat{\phi}} \tag{2}$$

$$\hat{a}^\dagger = e^{-i\hat{\phi}} \sqrt{\hat{n}+1}. \tag{3}$$

(b) Use the previous two equations to show that $\left[e^{i\hat{\phi}}, \hat{n}\right] = e^{i\hat{\phi}}$.

(c) Show that $\left[e^{i\hat{\phi}}, \hat{n}\right] = e^{i\hat{\phi}}$ holds if $\left[\hat{\phi}, \hat{n}\right] = -i$. To do so, expand the exponentials and use induction to prove $i\left[\hat{\phi}^k, \hat{n}\right] = k\hat{\phi}^{k-1}$ for $k \in \mathbb{N}$, assuming $\left[\hat{\phi}, \hat{n}\right] = -i$.

This commutator relation between operators $\hat{\phi}$ and $\hat{n}$ yields the Heisenberg-type uncertainty relation

$$\Delta\hat{\phi} \cdot \Delta\hat{n} \geq \frac{1}{2}. \tag{4}$$

(d) The previous results suggest that number and phase behave like canonically conjugate observables (similar to position and momentum): A higher degree of certainty in one of the two has to be paid for by more uncertainty in the other one. This reasoning is sometimes used in discussing phase diagrams, e.g. of squeezed states.

There is a problem, however. Imagine a case where the uncertainty in the particle number is small, say $\Delta\hat{n} < \frac{1}{4\pi}$. What does this imply for $\Delta\hat{\phi}$? Does the result make sense?

(e) Even worse, consider matrix elements of the commutator $\left[\hat{\phi}, \hat{n}\right] = -i$ between different Fock states $|m\rangle$, $|m'\rangle$. Show that one obtains an obviously wrong result if $m = m'$.

(f) The problems with the phase operator encountered here can be traced back to the fact that, despite their appearance, the operators $e^{i\hat{\phi}}$, $e^{-i\hat{\phi}}$ are not unitary, and in turn $\hat{\phi}$ is not Hermitian. Show explicitly that $\hat{\eta} = e^{i\hat{\phi}}$ is not unitary. To this end, note that $\hat{\eta} = \frac{1}{\sqrt{\hat{n}+1}}\hat{a}$, and check that $\hat{\eta}\hat{\eta}^\dagger = 1$. Now show that the expectation value of $\hat{\eta}^\dagger\hat{\eta}$ in the vacuum state is 0, and therefore $\hat{\eta}^\dagger\hat{\eta} \neq 1$.

Following Dirac's work, there has been much interest in defining more appropriate phase operators, for a review see e.g. M.M. Nieto, *Quantum Phase and Quantum Phase Operators: Some Physics and Some History*, `http://arxiv.org/abs/hep-th/9304036`.

7 Points

(please turn the page)

Exercise 8. Glauber-Sudarshan P-Distribution

Show that the density operator $\hat{\rho}$ can be expressed as

$$\hat{\rho} = \int \mathrm{d}^2\alpha P(\alpha)|\alpha\rangle\langle\alpha|,$$

where $P(\alpha) = \frac{1}{\pi}\sum_{k,l} \rho_{kl}^{(a)} \alpha^k \alpha^{*j}$ is the P-distribution you know from the lectures, and $\{|\alpha\rangle\}$ are the coherent states.

3 Points

Paul A. M. Dirac

Paul Adrien Maurice Dirac was born on 8th August, 1902, at Bristol, England, his father being Swiss and his mother English. He was educated at the Merchant Venturer's Secondary School, Bristol, then went on to Bristol University. Here, he studied electrical engineering, obtaining the B.Sc. (Engineering) degree in 1921. He then studied mathematics for two years at Bristol University, later going on to St. John's College, Cambridge, as a research student in mathematics. He received his Ph.D. degree in 1926. The following year he became a Fellow of St.John's College and, in 1932, Lucasian Professor of Mathematics at Cambridge.

Dirac's work has been concerned with the mathematical and theoretical aspects of quantum mechanics. He began work on the new quantum mechanics as soon as it was introduced by Heisenberg in 1925 - independently producing a mathematical equivalent which consisted essentially of a non-commutative algebra for calculating atomic properties - and wrote a series of papers on the subject, published mainly in the Proceedings of the Royal Society, leading up to his relativistic theory of the electron (1928) and the theory of holes (1930). This latter theory required the existence of a positive particle having the same mass and charge as the known (negative) electron. This, the positron was discovered experimentally at a later date (1932) by C. D. Anderson, while its existence was likewise proved by Blackett and Occhialini (1933) in the phenomena of "pair production" and "annihilation".

The importance of Dirac's work lies essentially in his famous wave equation, which introduced special relativity into Schrödinger's equation. Taking into account the fact that, mathematically speaking, relativity theory and quantum theory are not only distinct from each other, but also oppose each other, Dirac's work could be considered a fruitful reconciliation between the two theories.

Dirac's publications include the books Quantum Theory of the Electron (1928) and The Principles of Quantum Mechanics (1930; 3rd ed. 1947). He was elected a Fellow of the Royal Society in 1930, being awarded the Society's Royal Medal and the Copley Medal. He was elected a member of the Pontifical Academy of Sciences in 1961.

Dirac has travelled extensively and studied at various foreign universities, including Copenhagen, Göttingen, Leyden, Wisconsin, Michigan, and Princeton (in 1934, as Visiting Professor). In 1929, after having spent five months in America, he went round the world, visiting Japan together with Heisenberg, and then returned across Siberia. In 1937 he married Margit Wigner, of Budapest. Paul A.M. Dirac died on October 20, 1984.

(Text and picture taken from
`http://nobelprize.org/nobel_prizes/physics/laureates/1933/dirac-bio.html`)