Winter term 2012/13

Exercise Sheet 2, Theoretical Quantum and Atom Optics

University of Hamburg, Prof. P. Schmelcher

To be returned on Tuesday, 06/11/2012, in the tutorials

Exercise 3. Black-body radiation

We study the radiation in a closed cavity at temperature T. Each individual mode can be considered as a harmonic oscillator of angular frequency ω with energy

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega, \quad n \geq 0.$$

In the following we focus on a single radiation mode. The probability that there will be n photons in the mode is given by Boltzmann's law

$$P_\omega(n) = \frac{\exp(-E_n/k_B T)}{\sum_{n=0}^{\infty}\exp(-E_n/k_B T)}.$$

Show that $P_\omega(n)$ can be written as

$$P_\omega(n) = \frac{1}{\bar{n}+1}\left(\frac{\bar{n}}{\bar{n}+1}\right)^n,$$

where $\bar{n}$ is the average photon number.

Hints:

(i) Limit of the geometric series:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad \text{for } |x| < 1.$$

(ii) Average photon number is given by:

$$\bar{n} = \sum_{n=0}^{\infty} n\, P_\omega(n)$$

5 Points

Exercise 4. Coherent states

In the lectures it was stated that a coherent state of the quantum harmonic oscillator can be obtained by shifting the ground state in space by an arbitrary distance q_0.

- Using the definition of the ladder operators $\hat{a}$, $\hat{a}^\dagger$ in terms of position and momentum operators, show that for $\alpha \in \mathbb{R}$ the displacement operator $\hat{D}(\alpha)$ can be identified with the spatial translation operator $\hat{T}(q_0)$.

  To this end, remember that the momentum operator generates translations in quantum mechanics:

  $$\hat{T}(q_0) = e^{-\frac{i}{\hbar}q_0\hat{p}}$$

  What is the relation between α and q_0 if $\hat{D}(\alpha) = \hat{T}(q_0)$?

- Now you know that shifting the "vacuum" (i.e., the oscillator ground state) in position space produces a coherent state. In the lectures you have seen that evolving such a shifted ground state in time leads to a Gaussian wave packet oscillating in the trap and retaining its shape. You are now asked to fill in the missing steps in the corresponding calculation.

  Assume that at time $t = 0$ you start with the oscillator ground state, displaced by q_0:

  $$\psi(q,0) = \left(\frac{\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{\omega}{2\hbar}(q-q_0)^2\right).$$

  As in the lectures, the mass m has been set to 1.

To study this state's time evolution, expand it in terms of harmonic oscillator eigenfunctions: $\psi(q,0) = \sum_{k=0}^{\infty} c_k \phi_k(q)$. Show that

$$c_k = (2^k k!)^{-1/2} x_0^k \exp(-\frac{x_0^2}{4})$$

where $x_0 = \sqrt{\frac{\omega}{\hbar}} q_0$.

Now determine $\psi(q,t)$ and the time-dependent probability density $|\psi(q,t)|^2$. You should recover the result presented in the lectures.

Hints: It is useful to introduce the scaled variable $x = \sqrt{\frac{\omega}{\hbar}} q$. The harmonic oscillator eigenfunctions are given by

$$\phi_n(x) = (2^n n!)^{-1/2} \left(\frac{\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{x^2}{2}\right) H_n(x)$$

where $H_n(x)$ denotes the n-th Hermite polynomial.

You may use the following identities involving the H_k:

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-(x-y)^2} H_k(x) \mathrm{d}x &= \sqrt{\pi} 2^k y^k, \qquad \text{for } y \in \mathbb{R} \\ \sum_{k=0}^{\infty} \frac{s^k}{k!} H_k(x) &= e^{-s^2+2sx}, \qquad \text{for } s \in \mathbb{C}. \end{aligned}$$

5 Points