SS 2012

**Excercise Sheet 11, Lecture Theoretical Quantum and Atom Optics**

University of Hamburg, Prof. P. Schmelcher

To be returned on 02/07/2012, in the tutorials

**Exercise 22.** Bogoliubov Equations

Start from the time-dependent Gross-Pitaevskii equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi(r,t)+V(r)\psi(r,t)+U_0|\psi(r,t)|^2\psi(r,t)=i\hbar\frac{\partial\psi(r,t)}{\partial t}, \tag{1}$$

and assume that it has an unperturbed solution $\psi_0(\mathbf{r},t)=\phi_0(\mathbf{r})e^{-i\mu t/\hbar}$.
(i) Derive the linearized equation for a perturbation $\delta\psi(\mathbf{r},t)$ around this solution. To do so, assume that the wavefunction of the condensate only slightly deviates from the unperturbed state, $\psi(\mathbf{r},t)=\psi_0(\mathbf{r},t)+\delta\psi(\mathbf{r},t)$, and linearize the GP equation in terms of this small deviation.

(ii) Search the solutions $\delta\psi(\mathbf{r},t)$ of the linearized equation with the Bogoliubov Ansatz

$$\delta\psi(\mathbf{r},t)=e^{-i\mu t/\hbar}[u(\mathbf{r})e^{-i\omega t}-v^*(\mathbf{r})e^{i\omega^* t}].$$

Derive the Bogoliubov equations for $u(\mathbf{r}),v(\mathbf{r})$.

(iii) Assuming a uniform Bose gas, $V(\mathbf{r})=0$, and its unperturbed homogeneous solution $\psi(\mathbf{r},t)=\sqrt{n(\mathbf{r})}e^{-i\mu t/\hbar}$, where the chemical potential is given by $\mu=nU_0$.
Solutions of the Bogoliubov equations can now be sought in the form of plane waves $u(\mathbf{r})=u_q\frac{e^{i\mathbf{qr}}}{\sqrt{\mathcal{V}}}$, $v(\mathbf{r})=v_q\frac{e^{i\mathbf{qr}}}{\sqrt{\mathcal{V}}}$, where $\mathcal{V}$ is the volume of the system and $u_q,v_q$ are constant coefficients. Show that the Bogoliubov equations can then be written in the form

$$\left(\frac{\hbar^2q^2}{2m}+nU_0-\hbar\omega\right)u_q-nU_0v_q=0 \tag{2}$$

$$\left(\frac{\hbar^2q^2}{2m}+nU_0+\hbar\omega\right)v_q-nU_0u_q=0 \tag{3}$$

Find the dispersion relation $\omega(q)$ by studying the solvability condition for this homogeneous linear system. Show that $\omega(q)$ is asymptotically linear for small $q$ and quadratic for large $q$. Convince yourself that for attractive interactions, $U_0<0$, the Bogoliubov spectrum contains imaginary modes, signalling instability towards collapse.

**5 Points**

**Exercise 23.** Dark solitons

The repulsive ($U_0>0$), uniform ($V(\mathbf{r})=0$), one-dimensional Gross-Pitaevskii equation posseses the solitonic solution

$$\psi(x,t)=\sqrt{n_0}\left[i\frac{u}{s}+\sqrt{1-u^2/s^2}\tanh\left(\frac{x-ut}{\sqrt{2}\xi_u}\right)\right]e^{-i\mu t/\hbar}, \tag{4}$$

where $\xi=\hbar/\sqrt{2mn_0U_0}$ is the so-called coherence length, $\xi_u=\xi/\sqrt{1-u^2/s^2}$, and $s=\sqrt{\frac{n_0U_0}{m}}$ is the sound velocity in the uniform gas with density $n_0$.
The parameter $u<s$ denotes the soliton velocity.

(i) Find the minimum density $n_{min}$ of this object.

(ii) Show that the soliton velocity is equal to the bulk sound velocity evaluated at the density $n_{min}$, i.e., to the sound velocity of a uniform gas with density $n_{min}$.

(iii) Show that the total phase change $\varphi(x\to+\infty)-\varphi(x\to-\infty)$ (where $\varphi$ denotes the argument of $\psi$) across the soliton at fixed time is monotonically related to its velocity and is equal to $-2\arccos(u/s)$.

**5 Points**