

Lectures on classical optics

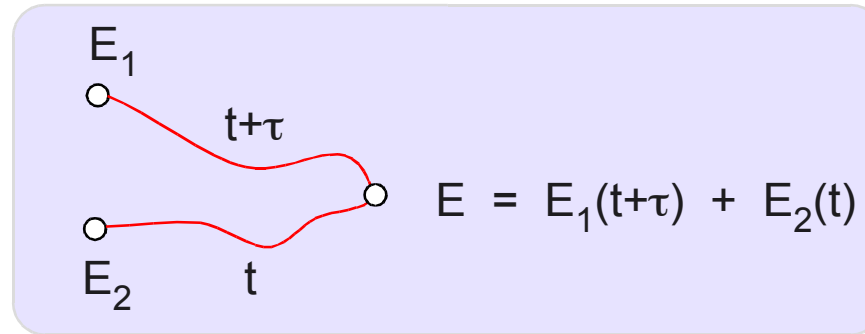
Part III

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The concept of coherence

consider harmonic fields E_1, E_2 at positions r_1, r_2 at time $t=0$:



$$\langle I_n \rangle = \langle E_n(t)E_n^*(t) \rangle, \quad n \in \{1,2\}$$

$$\langle I \rangle = \langle EE^* \rangle = \langle I_1 \rangle + \langle I_2 \rangle + 2 \operatorname{Re}[\langle E_1(t+\tau)E_2^*(t) \rangle] \quad \text{where} \quad \langle f \rangle \equiv \lim_{T \rightarrow \infty} \langle f \rangle_T, \quad \langle f \rangle_T \equiv \frac{1}{T} \int_{-T/2}^{T/2} f(t) dt$$

$\lim_{T \rightarrow \infty}$ means T finite but sufficiently large such that $\langle f \rangle_T$ does not depend on T

Normalized pair correlation function:
$$\gamma_{12}(\tau) \equiv \frac{\langle E_1(t+\tau)E_2^*(t) \rangle}{(\langle I_1 \rangle \langle I_2 \rangle)^{1/2}}$$

$$\Rightarrow \langle I \rangle = \langle I_1 \rangle + \langle I_2 \rangle + 2 (\langle I_1 \rangle \langle I_2 \rangle)^{1/2} \operatorname{Re}[\gamma_{12}(\tau)]$$

$$\gamma_{12}(\tau) = |\gamma_{12}(\tau)| \exp(i\phi_{12}(\tau)) \Rightarrow \langle I \rangle = \langle I_1 \rangle + \langle I_2 \rangle + 2 (\langle I_1 \rangle \langle I_2 \rangle)^{1/2} |\gamma_{12}(\tau)| \cos(\phi_{12}(\tau))$$

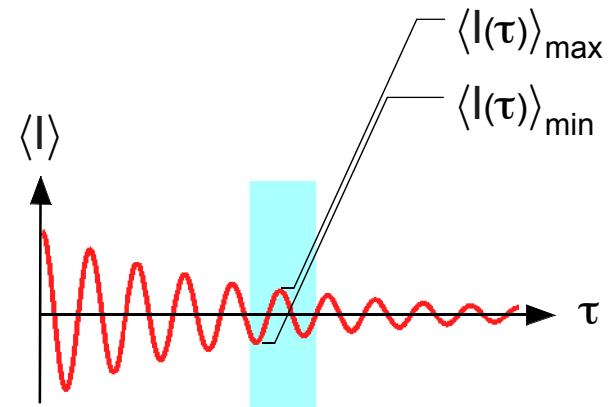
Assume: $|\gamma_{12}(\tau)|$ changes much slower than $\phi_{12}(\tau)$ (weakly coherent light)

$$\Rightarrow \langle I \rangle_{\max/\min} = \langle I_1 \rangle + \langle I_2 \rangle \pm 2 (\langle I_1 \rangle \langle I_2 \rangle)^{1/2} |\gamma_{12}(\tau)|$$

Interference visibility:

$$\kappa \equiv \left| \frac{\langle I \rangle_{\max} - \langle I \rangle_{\min}}{\langle I \rangle_{\max} + \langle I \rangle_{\min}} \right| = \frac{2 (\langle I_1 \rangle \langle I_2 \rangle)^{1/2}}{(\langle I_1 \rangle + \langle I_2 \rangle)} |\gamma_{12}(\tau)|$$

$$\langle I_1 \rangle = \langle I_2 \rangle \Rightarrow \kappa(\tau) = |\gamma_{12}(\tau)|$$

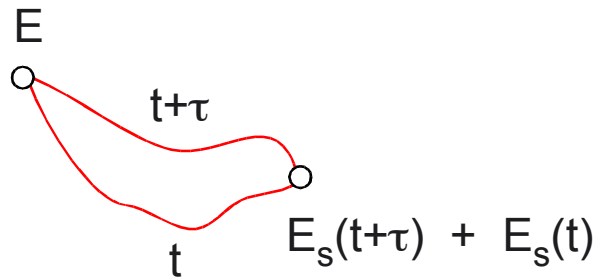


Definition: $|\gamma_{12}(\tau)| = 1$ for all τ \Rightarrow complete coherence

$0 < |\gamma_{12}(\tau)| < 1$ for some τ \Rightarrow partial coherence

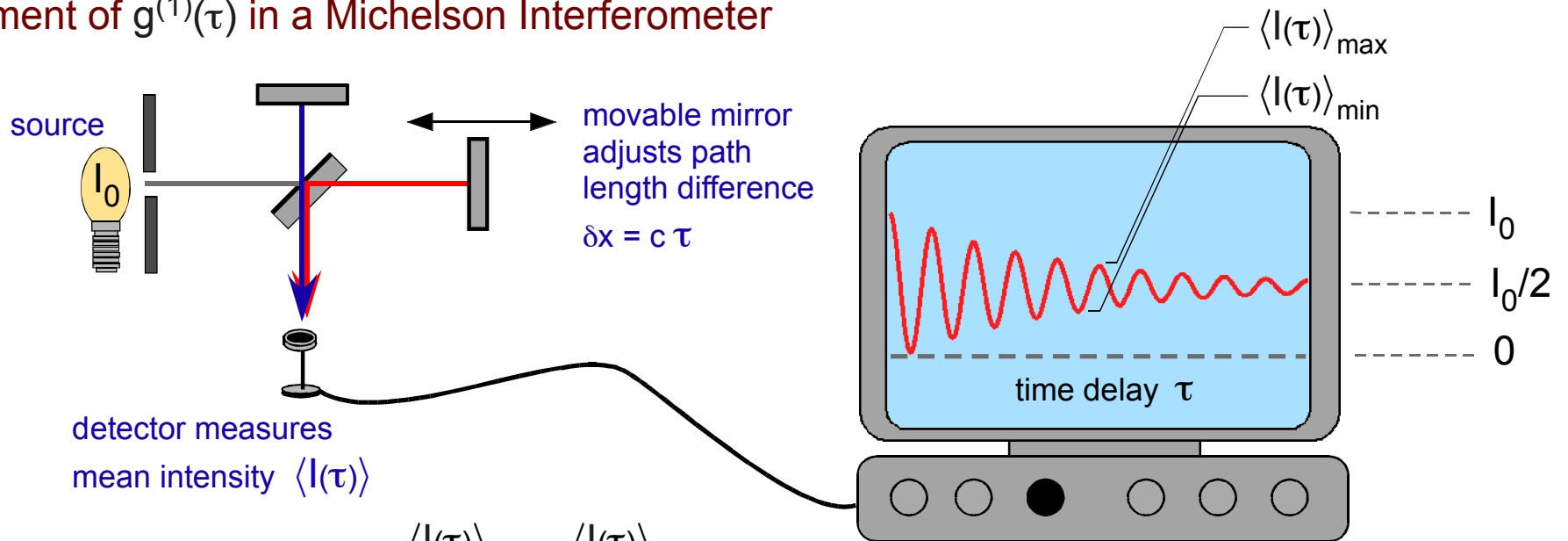
$|\gamma_{12}(\tau)| = 0$ for all τ \Rightarrow no coherence

Normalized autocorrelation function: $g^{(1)}(\tau) \equiv \frac{\langle E(t+\tau)E^*(t) \rangle}{\langle I \rangle}$ degree of first order coherence



$g^{(1)} : \mathbb{R} \rightarrow \{c \in \mathbb{C} : |c| \leq 1\}$ with $g^{(1)}(0) = 1, g^{(1)}(-\tau) = g^{(1)*}(\tau)$

measurement of $g^{(1)}(\tau)$ in a Michelson Interferometer



Interference contrast: $\kappa(\tau) \equiv \frac{\langle I(\tau) \rangle_{\max} - \langle I(\tau) \rangle_{\min}}{\langle I(\tau) \rangle_{\max} + \langle I(\tau) \rangle_{\min}}$

maximal coherence: Interference contrast maximal for all τ



partial coherence: Interference contrast decreases for large τ



Normalized autocorrelation function:

$$g^{(1)}(\tau) \equiv \frac{\langle E(t+\tau)E^*(t) \rangle}{\langle I \rangle}$$

Example: successive wave trains
of duration τ_0 and length $c \tau_0$

$$E(t) = E_0 \exp[i\omega t + i\phi(t)] \quad \text{with} \quad \phi(t):$$

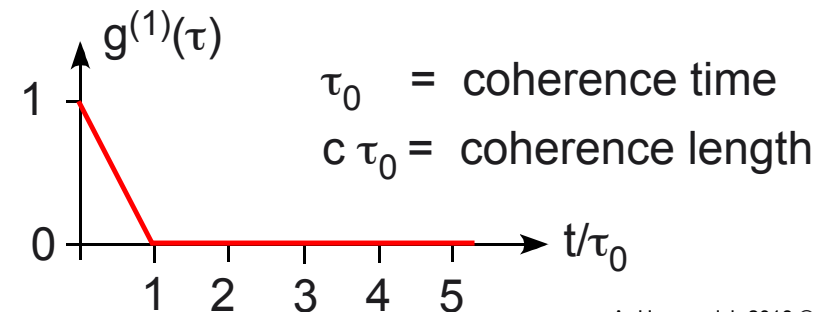
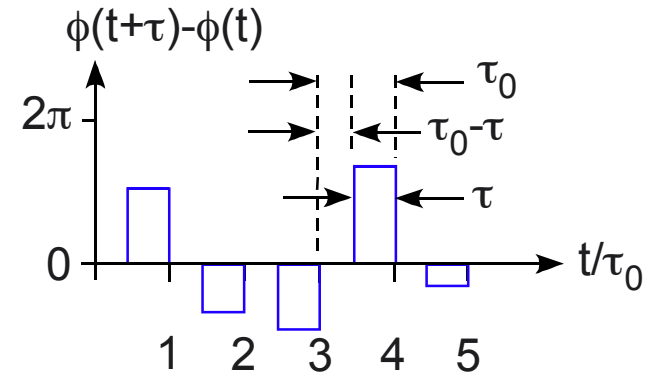
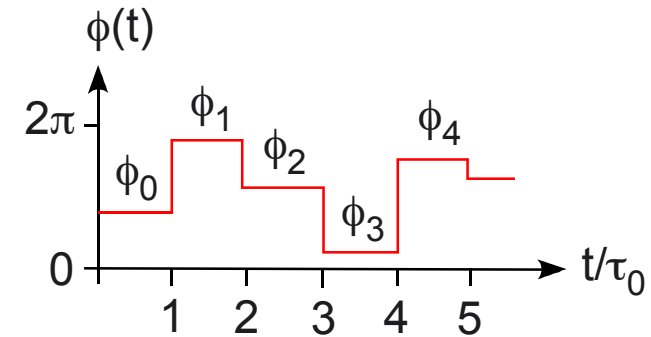
$$\Rightarrow g^{(1)}(\tau) = e^{i\omega\tau} \langle e^{i(\phi(t+\tau) - \phi(t))} \rangle$$

$0 \leq \tau \leq \tau_0$:

$$\langle e^{i(\phi(t+\tau) - \phi(t))} \rangle = \frac{1}{N\tau_0} \sum_{n=0}^{N-1} \int_{n\tau_0}^{(n+1)\tau_0} dt e^{i(\phi(t+\tau) - \phi(t))}$$

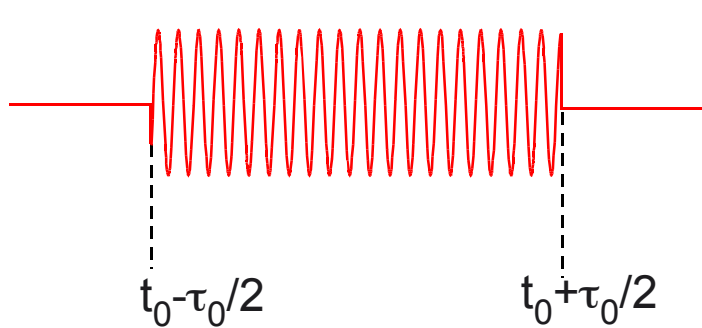
$$= \frac{1}{N\tau_0} \sum_{n=0}^{N-1} \left((\tau_0 - \tau) + \tau \exp(i(\phi_{n+1} - \phi_n)) \right)$$

$$\Rightarrow g^{(1)}(\tau) = e^{i\omega\tau} (\tau_0 - \tau)/\tau_0 \times \begin{cases} 0 & \text{if } \tau_0 < \tau \\ 1 & \text{if } 0 \leq \tau \leq \tau_0 \end{cases}$$



Coherence and emission spectrum:

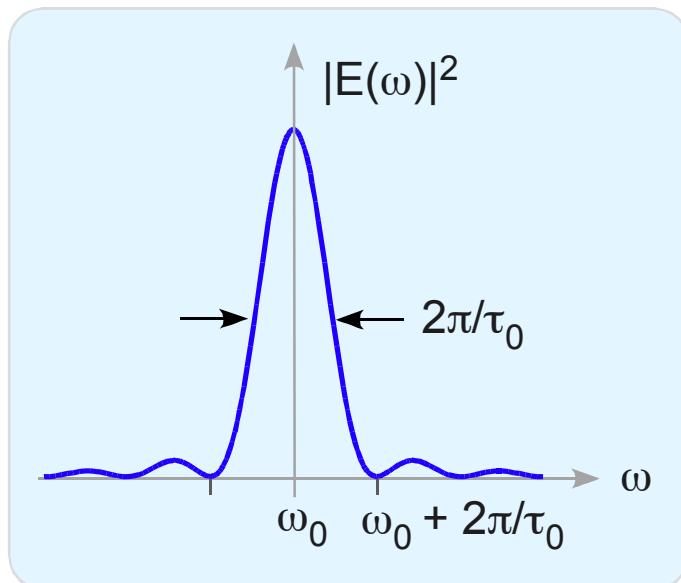
consider single wave train of duration τ_0 , phase ϕ_0 , frequency ω_0



$$E(t) = \exp[-i\omega_0 t - i\phi_0] \times \begin{cases} 0 & \text{otherwise} \\ 1 & \text{if } t_0 - \tau_0/2 \leq t \leq t_0 + \tau_0/2 \end{cases}$$

$$E(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dt E(t) e^{i\omega t} = \sqrt{\frac{2}{\pi}} \frac{\sin[(\omega - \omega_0)\tau_0/2]}{(\omega - \omega_0)} \exp(-i\phi_0)$$

N wave trains with the same frequency ω_0 but arbitrary phases ϕ_n , durations τ_n , starting times t_n :
(includes previous example of successive wave trains: $\tau_n = \tau_0$, $t_n = n \tau_0$)



$$E(\omega) = \sum_{n=1}^N \sqrt{\frac{2}{\pi}} \frac{\sin[(\omega - \omega_0)\tau_n/2]}{(\omega - \omega_0)} \exp(i(\omega - \omega_0)t_n - i\phi_n)$$

$$|E(\omega)|^2 \approx \sum_{n=1}^N |E_n(\omega)|^2 = \frac{2}{\pi} \sum_{n=1}^N \frac{\sin^2[(\omega - \omega_0)\tau_n/2]}{(\omega - \omega_0)^2}$$

Emission bandwidth $\Delta\nu \approx 1/\tau$ with $\tau \equiv \frac{1}{N} \sum_{n=1}^N \tau_n$

Wiener-Khintchine Theorem:

$$E(\omega) \equiv \mathcal{F}[E(t)] \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dt E(t) e^{i\omega t}, \quad F(\omega) \equiv \frac{|E(\omega)|^2}{\int_{-\infty}^{\infty} d\omega |E(\omega)|^2}$$

normalized spectral power density

$$\Rightarrow F(\omega) = \frac{1}{\sqrt{2\pi}} \mathcal{F}[g^{(1)}], \quad \mathcal{F} \equiv \text{Fourier-Transform}$$

Proof:
(for physicists)

$$\int_{-\infty}^{\infty} d\omega |E(\omega)|^2 = \int_{-\infty}^{\infty} dt |E(t)|^2 \approx \int_{-T/2}^{T/2} dt |E(t)|^2 \approx T \langle E(t)E^*(t) \rangle$$

$$|E(\omega)|^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dt dt' E(t') E^*(t) e^{i\omega(t'-t)}$$

$$\stackrel{t'=t+\tau}{=} \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dt d\tau E(t+\tau) E^*(t) e^{i\omega\tau} \approx \frac{T}{2\pi} \int_{-\infty}^{\infty} d\tau \langle E(t+\tau) E^*(t) \rangle e^{i\omega\tau}$$

$$F(\omega) \approx \frac{1}{2\pi} \int_{-\infty}^{\infty} d\tau e^{i\omega\tau} \frac{\langle E(t+\tau) E^*(t) \rangle}{\langle E(t) E^*(t) \rangle} = \frac{1}{\sqrt{2\pi}} \mathcal{F}[g^{(1)}]$$

Example: Collision broadened light source

Molecules of a gas radiate light $E(t) = E_0 \exp[-i(\omega_0 t + \phi(t))]$ at frequency ω_0 .

Collisions yield random phase jumps, i.e., phase $\phi(t) \in [0, 2\pi]$ fluctuates.

Probability for a free flight of duration $t \in [\tau, \tau + d\tau]$: $P(t) = \frac{1}{\tau_0} \exp(-\tau/\tau_0)$ (kinetic gas theory)

$\tau_0 =$ mean duration of a free flight

Coherence function: $g^{(1)}(\tau) = e^{i\omega_0 \tau} \langle e^{i(\phi(t+\tau) - \phi(t))} \rangle$

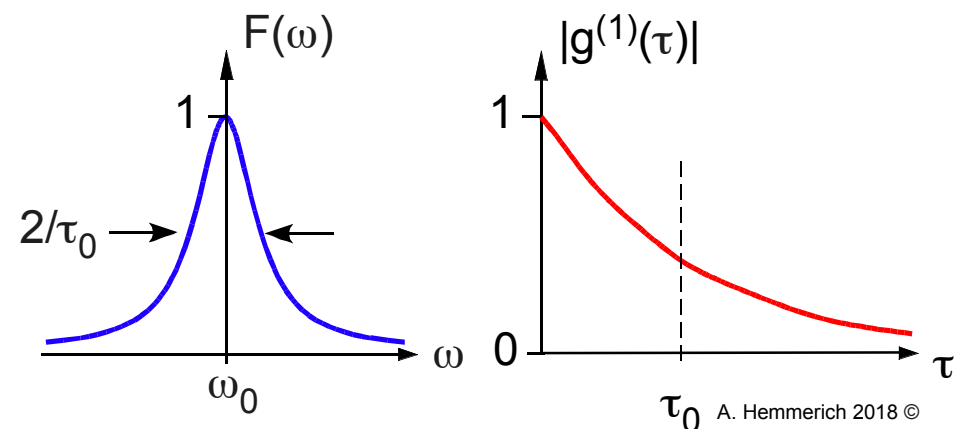
$$e^{i(\phi(t+\tau) - \phi(t))} = \begin{cases} 1 & \text{for free flights with duration } > \tau \\ e^{i\chi} \text{ with } \chi \in [0, 2\pi] \text{ random} & \text{for free flights with duration } < \tau \end{cases}$$

i.e., only flights of duration $t > \tau$ yield contribution to $\langle e^{i(\phi(t+\tau) - \phi(t))} \rangle$

$$\Rightarrow g^{(1)}(\tau) = e^{i\omega_0 \tau} \int_{\tau}^{\infty} P(s) ds = e^{i\omega_0 \tau} \exp(-\tau/\tau_0)$$

$$\Rightarrow |g(\tau)| = \exp(-\tau/\tau_0)$$

$$\Rightarrow F(\omega) \sim \frac{1}{1 + (\omega - \omega_0)^2 \tau_0^2}$$



Example: Doppler broadened light source

N Molecules of a gas radiate light $E_n(t) = E_0 \exp[i(\omega_0 t + \phi_n)]$ at frequency ω_0 in their restframe.

Motion of the molecules \Rightarrow Doppler-shift:

Probability for molecule with velocity $v \in [v, v+dv]$: $P(v)dv \sim \exp(-v^2/v_0^2) dv$ with $v_0 = (m/2k_B T)^{1/2}$

Probability for emission with frequency $\omega_0 + \delta$: $P(\delta) \sim \exp(-\delta^2/\delta_0^2)$ with $\delta_0 = (m/2k_B T)^{1/2} \omega/c$

Total field:
$$E(t) = E_0 \sum_{n=1}^N \exp[i(\omega_n t + \phi_n)], \quad \omega_n = \omega_0 + \delta_n$$

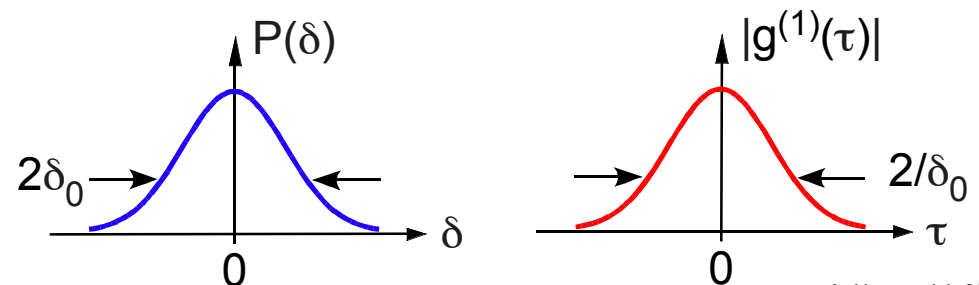
$$\langle E(t+\tau)E^*(t) \rangle = |E_0|^2 \sum_{n,m=1}^N \exp(i\omega_n \tau) \langle \exp[i(\omega_n - \omega_m)t] \exp[i(\phi_n - \phi_m)] \rangle$$

$$= |E_0|^2 \sum_{n=1}^N \exp(i\omega_n \tau) = |E_0|^2 \exp(i\omega_0 \tau) \sum_{n=1}^N \exp(i\delta_n \tau)$$

$$= N |E_0|^2 \exp(i\omega_0 \tau) \int_{-\infty}^{\infty} P(\delta) \exp(i\delta \tau) d\delta = N |E_0|^2 \exp(i\omega_0 \tau) \exp(-\delta_0^2 \tau^2)$$

$$g^{(1)}(\tau) = \exp(i\omega_0 \tau) \exp(-\delta_0^2 \tau^2)$$

(cf. W-K-Theorem)

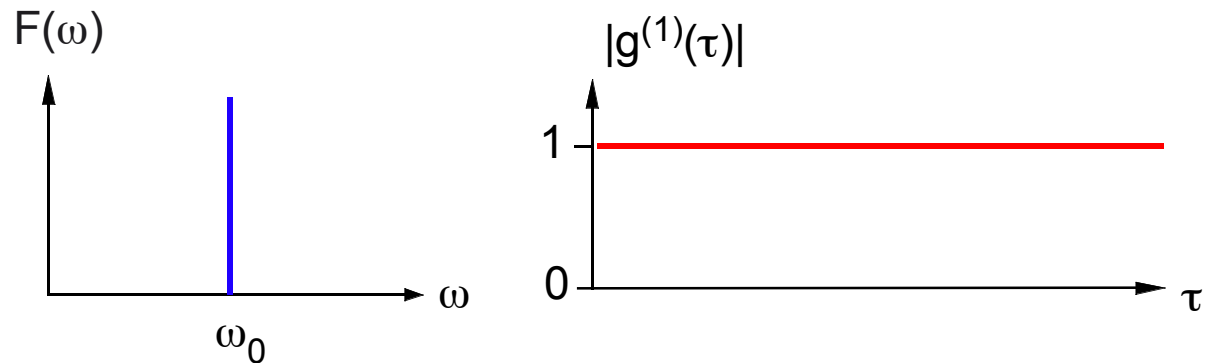


Example: monochromatic light

$$E(t) = E_0 \exp(i(\omega_0 t + \phi))$$

$$g^{(1)}(\tau) = \exp(i\omega_0 \tau)$$

$$|g^{(1)}(\tau)| = 1$$



Example: monochromatic light with frequency noise (Laser)

$$E(t) = E_0 \exp(i(\omega_0 t + \phi(t))), \quad \phi(t) = \int_0^t ds \Omega(s), \quad \Omega(s) = \text{frequency fluctuations}$$

$$\begin{aligned} (1) \quad \langle E(t+\tau)E^*(t) \rangle &= |E_0|^2 \exp(i\omega_0 \tau) \langle \exp[i(\phi(t+\tau) - \phi(t))] \rangle \\ &= |E_0|^2 \exp(i\omega_0 \tau) \exp\left[-\frac{1}{2} \langle [\phi(t+\tau) - \phi(t)]^2 \rangle\right] \end{aligned}$$

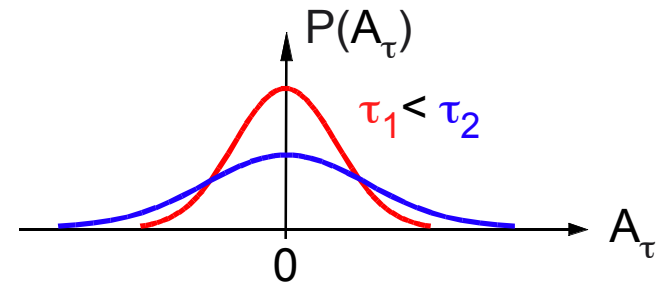
Gaussian moment theorem

Gaussian phase noise: evolution of $\phi(t)$ is an example of a Wiener-Lévy process

$$P(\phi_2, t+\tau | \phi_1, t) \equiv \frac{1}{\sqrt{2w \pi \tau}} \exp \left[\frac{-(\phi_2 - \phi_1)^2}{2w \tau} \right]$$

$$A_\tau(t) \equiv (\phi(t+\tau) - \phi(t)) \Rightarrow P(A_\tau) \sim \exp(-f(\tau) A_\tau^2)$$

$$\text{moments: } \langle A^{2n} \rangle \equiv \int_{-\infty}^{\infty} A^{2n} P(A) dA$$



Gaussian moment theorem:

$$A = \text{Gaussian variable} \Rightarrow \quad \text{(a)} \quad \langle A^{2n} \rangle = \frac{(2n)!}{2^n n!} \langle A^2 \rangle^n$$

$$\text{(b)} \quad \langle A^{2n+1} \rangle = 0$$

$$\langle \exp(iA) \rangle = \sum_{n=0}^{\infty} \frac{\langle (iA)^n \rangle}{n!} \stackrel{\text{(b)}}{=} \sum_{n=0}^{\infty} (-1)^n \frac{\langle A^{2n} \rangle}{(2n)!} \stackrel{\text{(a)}}{=} \sum_{n=0}^{\infty} (-1)^n \frac{\langle A^2 \rangle^n}{2^n n!} = \exp \left[-\frac{1}{2} \langle A^2 \rangle \right]$$

$$(2) \quad \langle [\phi(t+\tau) - \phi(t)]^2 \rangle = \left\langle \int_t^{t+\tau} \int_t^{t+\tau} ds ds' \Omega(s) \Omega(s') \right\rangle = \int_0^\tau \int_0^\tau ds ds' \langle \Omega(s+t) \Omega(s'+t) \rangle$$

$$\langle \Omega(s+t) \Omega(s'+t) \rangle = \langle \Omega(t) \Omega(s'-s+t) \rangle = \langle \Omega(s-s'+t) \Omega(t) \rangle$$

$$\Rightarrow \langle \Omega(s+t) \Omega(s'+t) \rangle = f(|s-s'|)$$

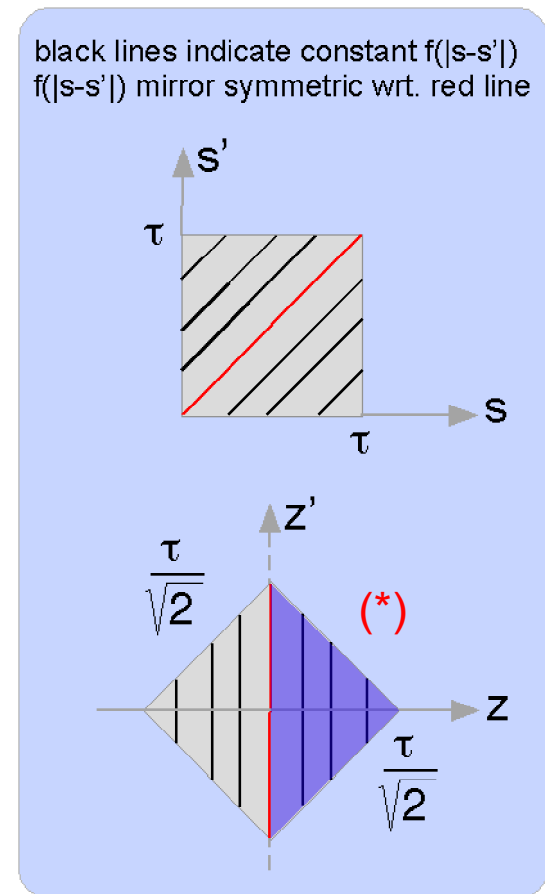
(3) coordinate transformation

$$\begin{pmatrix} z \\ z' \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} s \\ s' \end{pmatrix} - \frac{\tau}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \Rightarrow \quad dz dz' = ds ds'$$


$$\langle [\phi(t+\tau) - \phi(t)]^2 \rangle =$$

$$\int_0^\tau \int_0^\tau ds ds' f(|s-s'|) \stackrel{(*)}{=} 2 \int_0^\tau dz \int_{-(\tau/\sqrt{2}-z)}^{\tau/\sqrt{2}-z} dz' f(|\sqrt{2}z|)$$

$$= \sqrt{2} \int_0^\tau dx \int_{-((\tau-x)/\sqrt{2})}^{(\tau-x)/\sqrt{2}} dz' f(|x|) = 2 \int_0^\tau dx f(|x|) (\tau-x)$$



$$\begin{aligned}
\langle [\phi(t+\tau) - \phi(t)]^2 \rangle &= 2 \int_0^\tau ds \langle \Omega(s+t) \Omega(t) \rangle (\tau-s) = \frac{2}{T} \int_{-\infty}^{\infty} d\omega |\mathcal{F}[\Omega](\omega)|^2 \int_0^\tau ds (\tau-s) e^{-i\omega s} \\
&= \frac{2}{T} \int_{-\infty}^{\infty} d\omega |\mathcal{F}[\Omega](\omega)|^2 \left(\frac{2\sin^2(\omega\tau/2)}{\omega^2} + \frac{i}{\omega} \left(\frac{\sin(\omega\tau)}{\omega} - \tau \right) \right) \\
&= \frac{4}{T} \int_{-\infty}^{\infty} d\omega |\mathcal{F}[\Omega](\omega)|^2 \frac{\sin^2(\omega\tau/2)}{\omega^2} = \frac{4\tau}{T} \int_{-\infty}^{\infty} dz |\mathcal{F}[\Omega](z/\tau)|^2 \frac{\sin^2(z/2)}{z^2}
\end{aligned}$$


 Imaginary part is odd

$$\Rightarrow g^{(1)}(\tau) = \frac{\langle E(t+\tau)E^*(t) \rangle}{\langle E(t)E^*(t) \rangle} = \exp(i\omega_0\tau) \exp\left(-\frac{4\tau}{T} \int_{-\infty}^{\infty} dz |\mathcal{F}[\Omega](z/\tau)|^2 \frac{\sin^2(z/2)}{z^2}\right)$$

(1,2,3)

$$g^{(1)}(\tau) = \exp(i\omega_0\tau) \exp\left(-\frac{4\tau}{T} \int_{-\infty}^{\infty} dz |\mathcal{F}[\Omega](z/\tau)|^2 \frac{\sin^2(z/2)}{z^2}\right)$$

white frequency noise: $S \equiv \frac{1}{T} |\mathcal{F}[\Omega]|^2$ constant, $\tau_0^{-1} \equiv 2\pi S$

$$\int_{-\infty}^{\infty} \frac{\sin^2(z/2)}{z^2} dz = \frac{\pi}{2} \quad \Rightarrow \quad g^{(1)}(\tau) = \exp(i\omega_0\tau) \exp(-\tau/\tau_0)$$

W-K-Theorem \Rightarrow Lorentzian emission spectrum with $2/\tau_0$ FWHM

BSP für weißes Frequenzrauschen: Schalow Townes Quantenrauschen

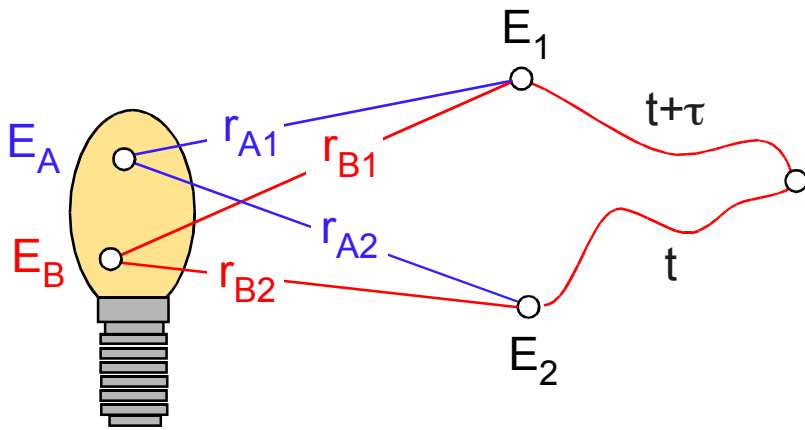
$$1/f \text{ frequency noise: } S \equiv \frac{1}{T} |\mathcal{F}[\Omega]|^2 = \frac{S_0}{|\omega|}, \quad S_0 \text{ constant, } \tau_0 \equiv \left(8S_0 \int_{0+}^{\infty} dz \frac{\sin^2(z/2)}{z^3}\right)^{-1/2}$$

(Note: Integral does not exist, if integration extends to zero!)

$$\Rightarrow g^{(1)}(\tau) = \exp(i\omega_0\tau) \exp[-(\tau/\tau_0)^2]$$

W-K-Theorem \Rightarrow Gaussian emission spectrum with $2/\tau_0$ 1/e-Full Width

Spatial Coherence



$$E_1 = E_{A1} + E_{B1}$$

$$E_2 = E_{A2} + E_{B2}$$

$$E_{An} = E_A \exp(i r_{An} \omega/c)$$

$$E_{Bn} = E_B \exp(i r_{Bn} \omega/c)$$

$$\langle E_1(t+\tau)E_2^*(t) \rangle = \langle E_{A1}(t+\tau)E_{A2}^*(t) \rangle + \langle E_{B1}(t+\tau)E_{B2}^*(t) \rangle$$

$$+ \cancel{\langle E_{A1}(t+\tau)E_{B2}^*(t) \rangle} + \cancel{\langle E_{B1}(t+\tau)E_{A2}^*(t) \rangle}$$

E_{An}, E_{Bn} mutually incoherent

$$\langle I_n \rangle = \langle E_n(t)E_n^*(t) \rangle = \langle E_{An}(t)E_{An}^*(t) \rangle + \langle E_{Bn}(t)E_{Bn}^*(t) \rangle$$

$$+ \cancel{\langle E_{An}(t)E_{Bn}^*(t) \rangle} + \cancel{\langle E_{Bn}(t)E_{An}^*(t) \rangle}$$

E_{An}, E_{Bn} mutually incoherent

$$\Rightarrow \langle I_1 \rangle = \langle I_2 \rangle$$

Light Source: consisting of mutually incoherent point sources with equal coherence properties described by $g^{(1)}(\tau) = \exp(i\omega\tau - \tau/\tau_0)$

$$E_{A_n} = E_A \exp(i r_{A_n} \omega/c) \Rightarrow$$

$$\langle E_{A_1}(t+\tau)E_{A_2}^*(t) \rangle = \langle E_A(t+\tau)E_A^*(t) \rangle \exp[i(r_{A_1} - r_{A_2})\omega/c] = \langle E_A(t+\tau_A)E_A^*(t) \rangle \quad \text{with } \tau_A \equiv \tau + (r_{A_1} - r_{A_2})/c$$

$$E_{B_n} = E_B \exp(i r_{B_n} \omega/c) \Rightarrow$$

$$\langle E_{B_1}(t+\tau)E_{B_2}^*(t) \rangle = \langle E_B(t+\tau)E_B^*(t) \rangle \exp[i(r_{B_1} - r_{B_2})\omega/c] = \langle E_B(t+\tau_B)E_B^*(t) \rangle \quad \text{with } \tau_B \equiv \tau + (r_{B_1} - r_{B_2})/c$$

$$\Rightarrow \langle E_1(t+\tau)E_2^*(t) \rangle = \langle E_{A_1}(t+\tau)E_{A_2}^*(t) \rangle + \langle E_{B_1}(t+\tau)E_{B_2}^*(t) \rangle = \langle E_A(t+\tau_A)E_A^*(t) \rangle + \langle E_B(t+\tau_B)E_B^*(t) \rangle$$

$$\langle I_n \rangle = \langle E_n(t)E_n^*(t) \rangle = \langle E_{A_n}(t)E_{A_n}^*(t) \rangle + \langle E_{B_n}(t)E_{B_n}^*(t) \rangle$$

$$\text{assume } \langle E_{A_n}(t)E_{A_n}^*(t) \rangle = \langle E_{B_n}(t)E_{B_n}^*(t) \rangle \text{ and hence } \langle I_n \rangle = 2\langle E_{A_n}(t)E_{A_n}^*(t) \rangle = 2\langle E_{B_n}(t)E_{B_n}^*(t) \rangle$$

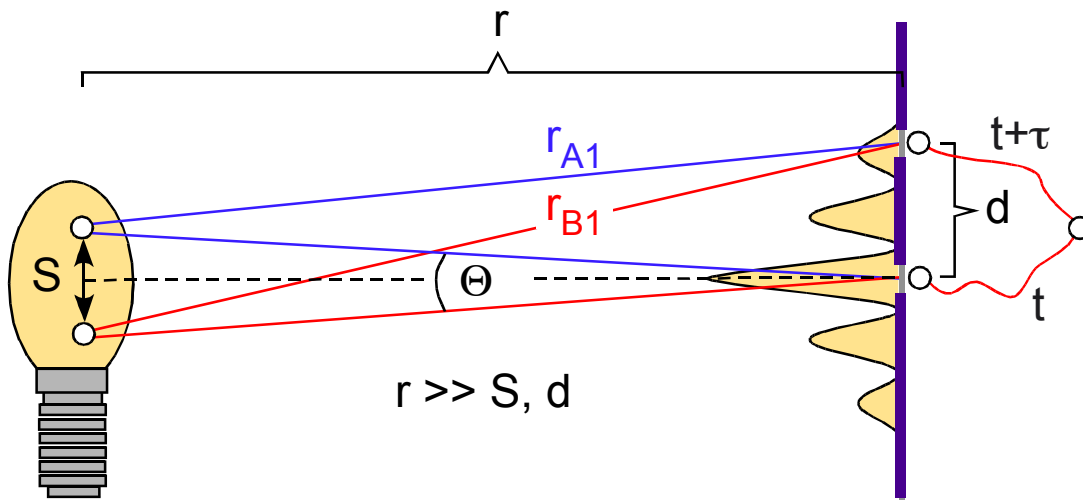
$$\gamma_{12}(\tau) \equiv \frac{\langle E_1(t+\tau)E_2^*(t) \rangle}{(\langle I_1 \rangle \langle I_2 \rangle)^{1/2}} = \frac{1}{2} [g^{(1)}(\tau_A) + g^{(1)}(\tau_B)] = \frac{1}{2} [\exp(i\omega\tau_A - \tau_A/\tau_0) + \exp(i\omega\tau_B - \tau_B/\tau_0)]$$

pair correlation is sum of $g^{(1)}$ -functions of each point source

$$4 |\gamma_{12}(\tau)|^2 = |g^{(1)}(\tau_A)|^2 + |g^{(1)}(\tau_B)|^2 + 2 |g^{(1)}(\tau_A)||g^{(1)}(\tau_B)| \cos(\omega(\tau_A - \tau_B)) \quad \text{Interference term !}$$

$$4 |\gamma_{12}(\tau)|^2 = |g^{(1)}(\tau_A)|^2 + |g^{(1)}(\tau_B)|^2 + 2 |g^{(1)}(\tau_A)||g^{(1)}(\tau_B)| \cos(\omega(\tau_A - \tau_B))$$

Interference term depends on $\tau_A - \tau_B = (r_{A1} - r_{A2})/c - (r_{B1} - r_{B2})/c$



Light Source: mutually incoherent point sources $g^{(1)}(\tau) = \exp(i\omega\tau - \tau/\tau_0)$

$$r_{A2} - r_{B2} = 0 \Rightarrow \tau_A - \tau_B = (r_{A1} - r_{B1})/c$$

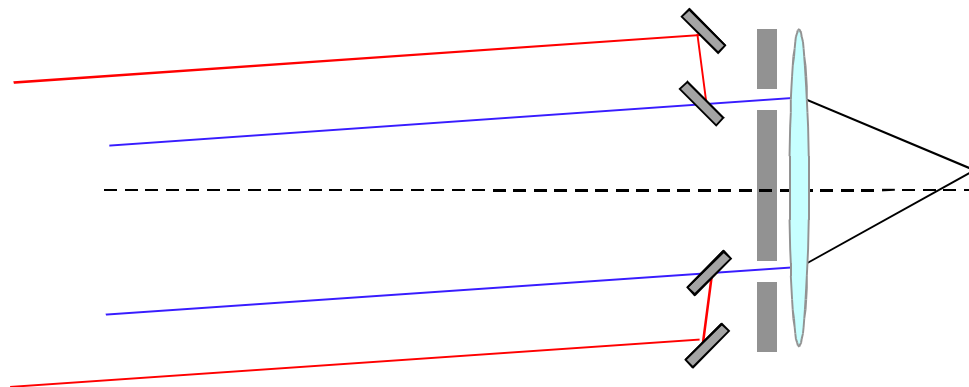
$$r_{A1} \approx r + \frac{(d-S/2)^2}{2r}, \quad r_{B1} \approx r + \frac{(d+S/2)^2}{2r}$$

$$\Rightarrow \tau_A - \tau_B \approx -\frac{S d}{2r c}$$

First minimum of $|\gamma_{12}(\tau)|^2$:

$$\omega(\tau_A - \tau_B) = \pi, \quad S \approx r \theta \Rightarrow d \approx \lambda/\theta$$

transverse coherence length



Michelson stellar interferometer: adjustable slits, extension of slit separation by mirrors

Measurement of angular diameter of stars, angular separation of double stars, etc.

Second Order Coherence

Normalized autocorrelation function: $g^{(2)}(\tau) \equiv \frac{\langle I(t+\tau)I(t) \rangle}{\langle I(t) \rangle^2}$ degree of second order coherence

(1) $g^{(2)} : \mathbb{R} \rightarrow \{r \in \mathbb{R} : r \geq 0\}$

(4) $g^{(2)}(\tau) \leq g^{(2)}(0)$

(2) $g^{(2)}(-\tau) = g^{(2)}(\tau)$

(5) $g^{(2)}(\tau \rightarrow \infty) = 1$ if correlations vanish

(3) $g^{(2)}(0) \geq 1$

Proof (3):

$$\left(\frac{1}{N} \sum_{n=1}^N I_n \right)^2 = \frac{1}{N^2} \left(\sum_n I_n^2 + \sum_{n \neq m} I_n I_m \right) \leq \frac{1}{N^2} \left(\sum_n I_n^2 + \sum_{n \neq m} (I_n^2 + I_m^2)/2 \right)$$

$$= \frac{1}{N^2} \sum_{n,m} (I_n^2 + I_m^2)/2 = \frac{1}{N} \sum_{n,m} I_n^2 \Rightarrow g^{(2)}(0) = \frac{\langle I(t)^2 \rangle}{\langle I(t) \rangle^2} = \frac{\frac{1}{N} \sum_{n,m} I_n^2}{\left(\frac{1}{N} \sum_{n=1}^N I_n \right)^2} \geq 1$$

Proof (4):

$$\left(\langle I(t+\tau)I(t) \rangle \right)^2 = \left(\frac{1}{N} \sum_{n=1}^N I(t_n+\tau)I(t_n) \right)^2 \leq \underbrace{\left(\frac{1}{N} \sum_{n=1}^N I(t_n+\tau)^2 \right) \left(\frac{1}{N} \sum_{n=1}^N I(t_n)^2 \right)}_{\text{Cauchy-Schwartz}} = \left(\langle I(t)^2 \rangle \right)^2$$

Proof (5): $\tau \rightarrow \infty \Rightarrow \langle I(t+\tau)I(t) \rangle = \langle I(t+\tau) \rangle \langle I(t) \rangle = \langle I(t) \rangle^2$
 no correlations

Example: monochromatic light

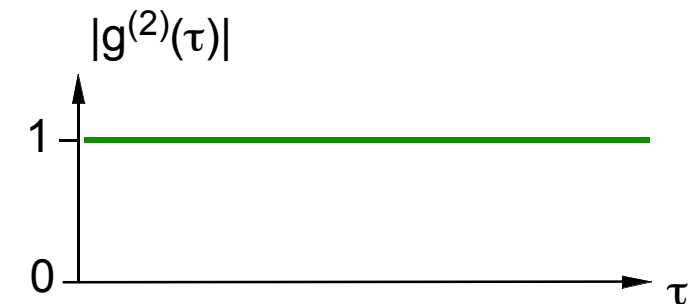
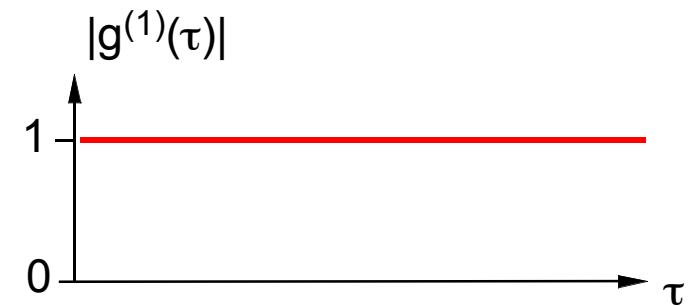
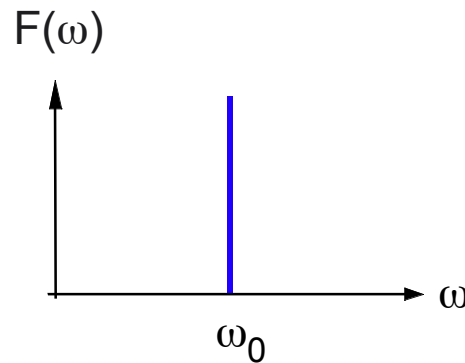
$$E(t) = E_0 \exp(i(\omega_0 t + \phi))$$

$$I(t) = E_0 E_0^*$$

$$g^{(1)}(\tau) = \exp(i\omega_0 \tau)$$

$$|g^{(1)}(\tau)| = 1$$

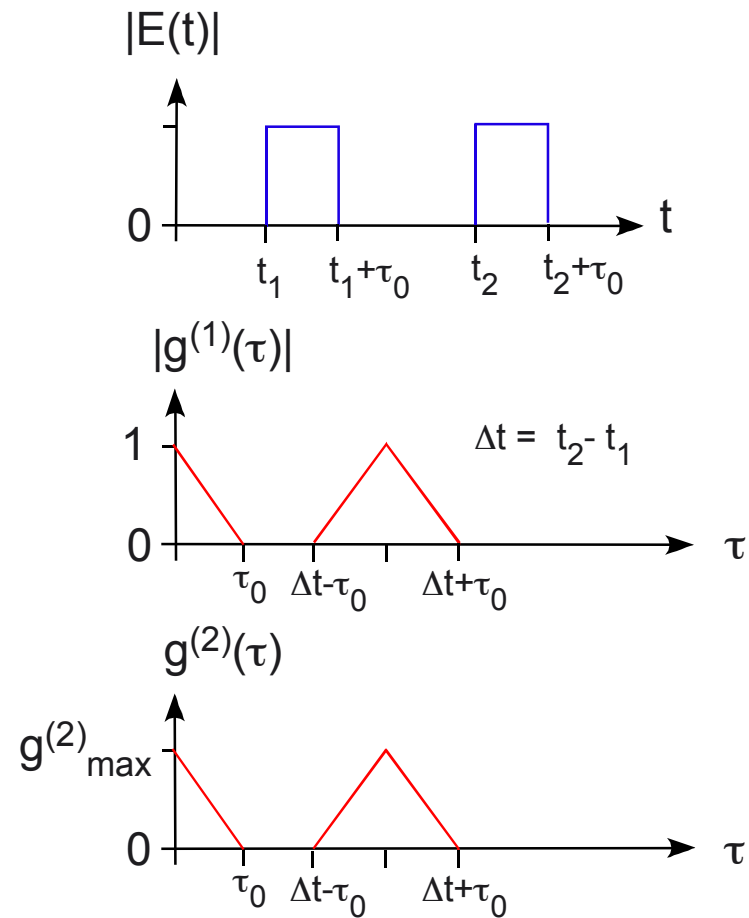
$$g^{(2)}(\tau) \equiv \frac{\langle I(t+\tau)I(t) \rangle}{\langle I(t) \rangle^2} = 1$$



Example: pulse train

$$g^{(2)}_{\max} = (N \tau_0)^{-1}$$

N = mean number of pulses/time



Gaussian chaotic Light:

Probability for $E(t) = A(t) \exp[i\phi(t)]$ to fall within unit area at the point (A, ϕ) in the complex plane C :

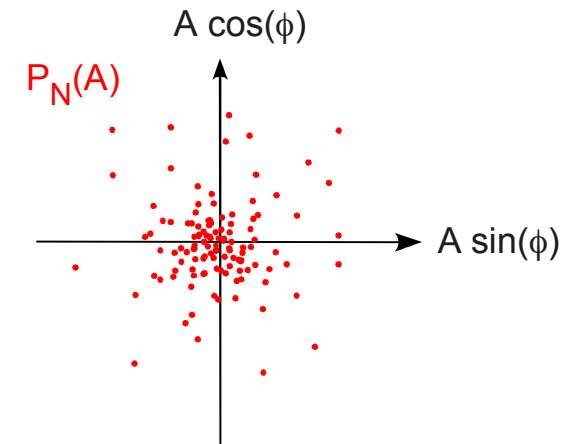
$$P_N(A, \phi) = \frac{1}{\pi N} \exp(-A^2/N)$$

Probability for measuring an intensity $\in [I, I + dI]$:

$$P(I) dI = \frac{1}{\langle I \rangle} \exp(-I / \langle I \rangle) dI$$

moments: $\langle I^n \rangle \equiv \int_0^{\infty} dI P(I) I^n = n! \langle I \rangle^n$

$$\Delta I \equiv (\langle I^2 \rangle - \langle I \rangle^2)^{1/2} = \langle I \rangle$$



Example:

$$E(t) = E_0 \sum_{n=1}^N \exp[i\phi_n(t)]$$

$\phi_n(t)$ = random phases, uniform at any time t , i.e., $P(\phi) = 1/2\pi$

$$\Rightarrow P_N(A, \phi) \approx \frac{1}{\pi N} \exp(-A^2/N)$$

NOTE: for chaotic light: $g^{(2)}(\tau) = 1 + |g^{(1)}(\tau)|^2$

$$E(t) = \sum_{n=1}^N E_n(t), \text{ with } E_n(t), E_m(t) \text{ uncorrelated for } n \neq m:$$

$$\Rightarrow \text{for } n \neq m \quad \langle E_n(t+\tau) E_m(t)^* \rangle = \langle E_n(t+\tau) \rangle \langle E_m(t)^* \rangle = 0$$

$$\begin{aligned} \langle E(t+\tau)E(t) E(t)^*E(t+\tau)^* \rangle &= \sum_{n,m,k,l} \langle E_n(t+\tau)E_m(t)^* E_k(t)E_l(t+\tau)^* \rangle \\ &= \sum_{n=1}^N \langle E_n(t+\tau)E_n(t) E_n(t)^*E_n(t+\tau)^* \rangle \\ &\quad + \sum_{n \neq m} \langle E_n(t+\tau)E_n(t)^* E_m(t)E_m(t+\tau)^* \rangle + \sum_{n \neq m} \langle E_n(t+\tau)E_n(t+\tau)^* E_m(t)^*E_m(t) \rangle \end{aligned}$$

$N \gg 1$, non-diagonal terms dominate

$$\approx \sum_{n \neq m} \langle E_n(t+\tau)E_n(t)^* \rangle \langle E_m(t)E_m(t+\tau)^* \rangle + \sum_{n \neq m} \langle E_n(t+\tau)E_n(t+\tau)^* \rangle \langle E_m(t)^*E_m(t) \rangle$$

$$\approx \left| \sum_{n=1}^N \langle E_n(t+\tau)E_n(t)^* \rangle \right|^2 + \left| \sum_{n=1}^N \langle E_n(t)^*E_n(t) \rangle \right|^2$$

$$\langle E(t+\tau) E(t)^* \rangle = \sum_{n,m}^N \langle E_n(t+\tau) E_m(t)^* \rangle = \sum_n^N \langle E_n(t+\tau) E_n(t)^* \rangle$$

$$\begin{aligned} \langle E(t+\tau) E(t) E(t)^* E(t+\tau)^* \rangle &= \left| \sum_{n=1}^N \langle E_n(t+\tau) E_n(t)^* \rangle \right|^2 + \left| \sum_{n=1}^N \langle E_n(t)^* E_n(t) \rangle \right|^2 \\ &= |\langle E(t+\tau) E(t)^* \rangle|^2 + |\langle E(t) E(t)^* \rangle|^2 \end{aligned}$$

$$\Rightarrow g^{(2)}(\tau) = \frac{\langle E(t+\tau) E(t) E(t)^* E(t+\tau)^* \rangle}{|\langle E(t) E(t)^* \rangle|^2} = \frac{|\langle E(t+\tau) E(t)^* \rangle|^2}{|\langle E(t) E(t)^* \rangle|^2} + 1 = |g^{(1)}(\tau)|^2 + 1$$

An example of chaotic light: collisional broadened source revisited

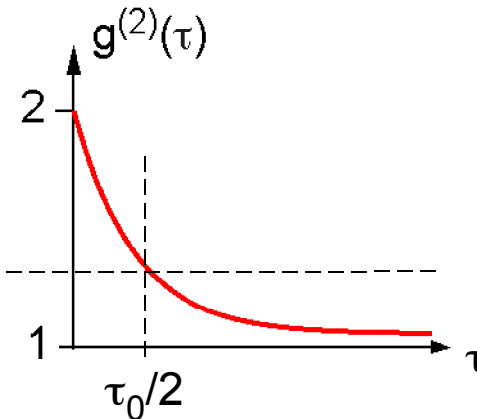
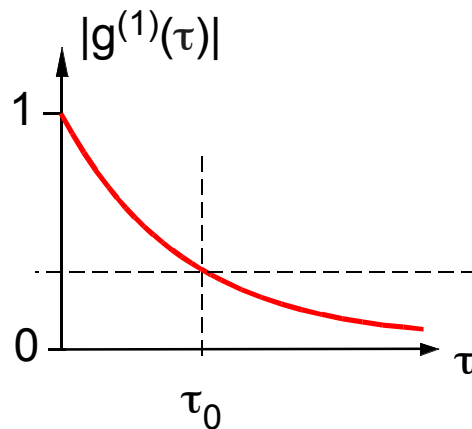
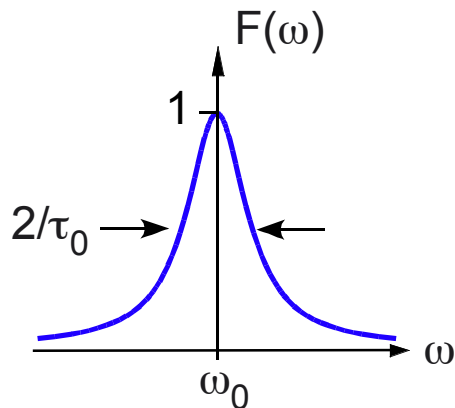
$$E(t) = E_0 \sum_{n=1}^N \exp[i\phi_n(t)], \quad \phi_n(t) = -\omega_n t + \xi_n, \quad \xi_n = \text{random phase} \quad \Rightarrow$$

$$g^{(1)}(\tau) = \sum_{n=1}^N \langle \exp[i(\phi_n(t+\tau) - \phi_n(t))] \rangle = \sum_{n=1}^N \langle \exp(i\omega_n \tau) \rangle = \int_{-\infty}^{\infty} d\omega \exp(i\omega\tau) P(\omega)$$

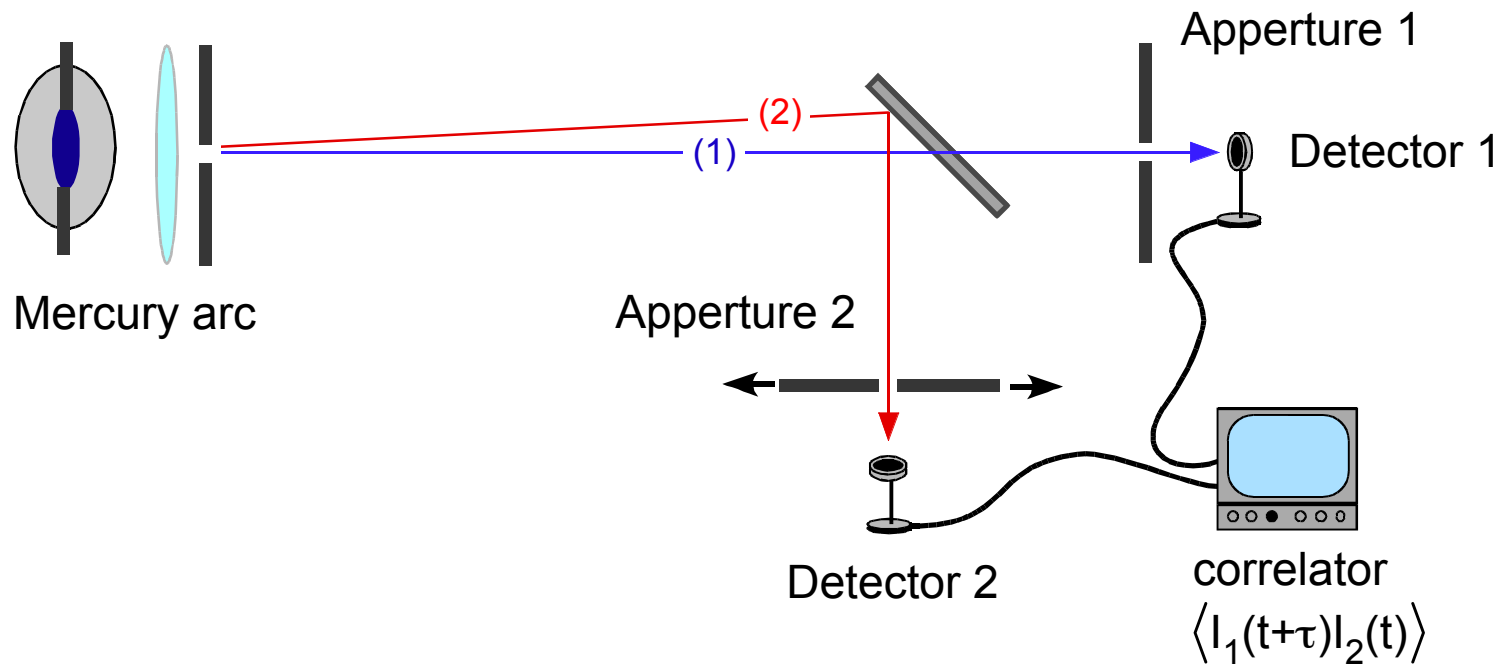
Example: assume Lorentzian spectrum (collision broadened light source)

$$P(\omega) = \frac{\tau_0}{\pi} \frac{1}{1 + \tau_0^2(\omega_0 - \omega)^2} \quad \Rightarrow \quad g^{(1)}(\tau) = \exp(-i\omega_0\tau - |\tau|/\tau_0)$$

$$g^{(2)}(\tau) = 1 + \exp(-2|\tau|/\tau_0)$$



Measurement of $g^{(2)}(\tau)$: Hanbury Brown & Twiss (1956)



Variation of aperture 2 permits measurement of transverse coherence length
→ determination of opening angle of source

R. Hanbury Brown, R. Q. Twiss, *Nature* 177,27 (1957)