Einführung in die Astronomie II
Teil 16

Peter Hauschildt
yeti@hs.uni-hamburg.de

Hamburger Sternwarte
Gojenbergsweg 112
21029 Hamburg

13. August 2019

Overview part 16

- Hitchiker's Guide to General Relativity
- Black Holes

Overview GR

- Einstein 1907–1915
- geometrical description of the effects of matter on space-time.
- uses *curved space-time* to describe motions under the effect of (conventional) gravity
- 2D representation:

Space-time curvature

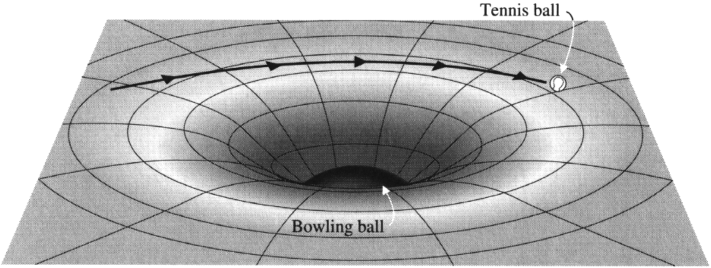


Figure 16.2 Rubber sheet analogy for curved space around the Sun.

Overview GR !!

- *mass acts on space-time, telling it how to curve*
- *curved space-time acts on mass, telling it how to move*
- explains part of Mercury's perihelion shift not accounted for by Newtonian mechanics
- light moves along the quickest route between two points
- curved space-time $\rightarrow$ bending of light rays

curved light paths

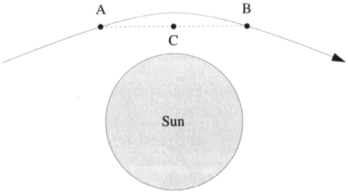


Figure 16.3 A photon's path around the Sun is shown by the solid line. The bend in the photon's trajectory is greatly exaggerated.

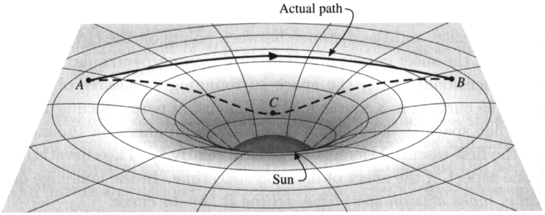


Figure 16.4 Comparison of two photon paths through curved space between points A and B.

Overview GR !!

- 2 cumulative effects:
  1. *length* of the path (shortest for light)
  2. *time dilation* (quickest for light)
- *time runs slower in curved space-time*
- this predicts a change in positions of stars if they are close to the Sun:

curved light paths

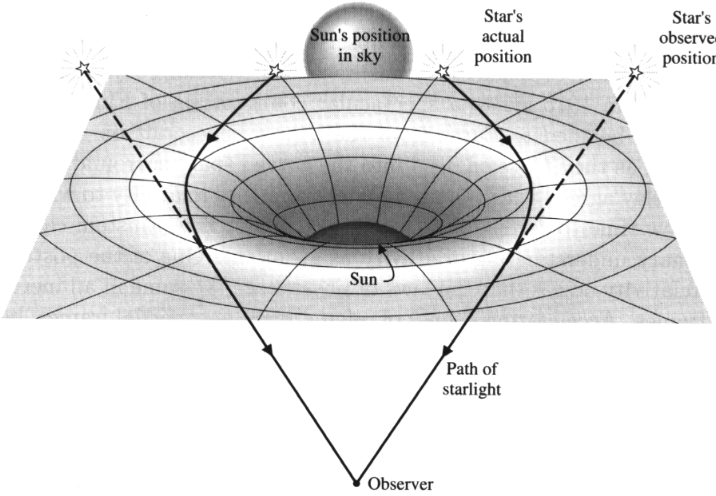


Figure 16.5 Bending of starlight measured during a solar eclipse.

Gravitational & Inertial Mass

- 2 particles, (m, q) and (M, Q) with mass and charge
- force due to gravity:

$$m_i a = G \frac{m_g M_g}{r^2}$$

- force due to electric field:

$$m_i a = \frac{qQ}{r^2}$$

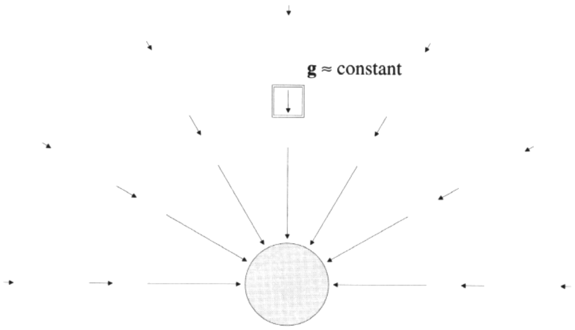
Gravitational & Inertial Mass !!

- inertial mass m_i: resistance to acceleration (left hand side)
- right hand side masses m_g: measure "charge" similar to electric charges
- experimental fact: $m_i/m_g = \mathrm{const.}$ (better than 10^{-12})
- consequence on Earth: everything falls with same acceleration
- chose units so that $m_i = m_g$ (changes G).

Principle of Equivalence !!

- special relativity $\rightarrow$ inertial frame motions
- gravity $\rightarrow$ accelerated, non-inertial reference frames
- idea: no gravity observed in free-falling coordinate system
- $\rightarrow$
  All local, freely falling, non-rotating laboratories are fully equivalent for the performance of all physical experiments
- *local inertial frames*

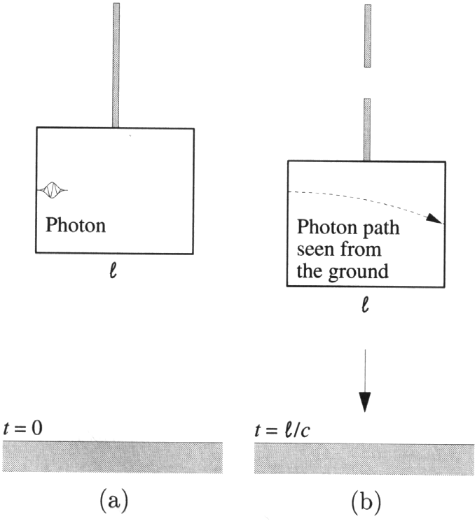
Principle of Equivalence



Principle of Equivalence

- special relativity is sub-set of general relativity
- Lorentz transformation with instantaneous velocity is used to transfer coordinates

curvature of space-time



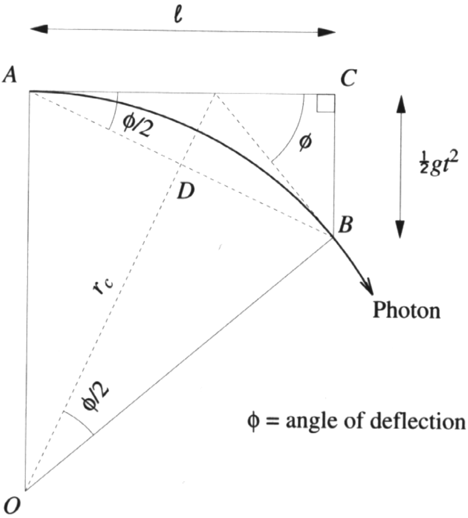
curvature of space-time

- "lab" suspended over ground
- photon emitted by light source when suspension severed
- lab is free falling $\rightarrow$ local inertial frame
- lab observer $\rightarrow$ light travels in straight horizontal line
- observer on ground $\rightarrow$ accelerated lab (gravity)
  $\rightarrow$ photon moves at constant distance from lab "floor"
  $\rightarrow$ photon path curved for observer on ground

curvature of space-time

- photon path is "quickest" route through
- $\rightarrow$ *curved space-time* around Earth
- estimating the angle of deflection:

curvature of space-time



curvature of space-time

- curved path approximated by circle
- ℓ: width of the lab (path length)
- arc length AB $\approx \ell$
- photon crossing time: $t = \ell / c$
- free fall distance $C \to B$: $d = \frac{1}{2} g t^2$

curvature of space-time

- from the geometry of the triangles:

$$\frac{\bar{BC}}{\bar{AC}} = \frac{\bar{BD}}{\bar{OD}}$$

or

$$\frac{\frac{1}{2}gt^2}{\ell} = \frac{\frac{\ell}{2\cos(\phi/2)}}{\bar{OD}}$$

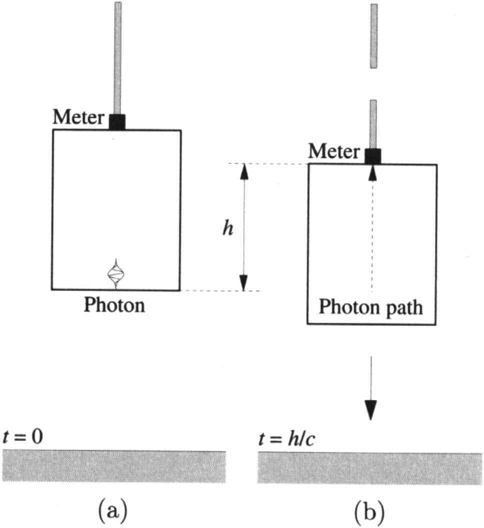
- $\phi \ll 1 \rightarrow \cos(\phi/2) \approx 1$
- $\bar{OD} \approx r_c \rightarrow$

$$r_c = \frac{c^2}{g}$$

curvature of space-time

- Earth: $r_c = 9.2 \times 10^{17}\,\mathrm{cm} \approx 0.2\,\mathrm{pc}$
- $\ell = 10\,\mathrm{m} \rightarrow \phi = \ell / r_c \approx 2.3 \times 10^{-10}$ arcsec

gravitational redshift



gravitational redshift

- same setup as previously, but light source emits photons vertically upward
- lab free falling local inertial frame
  $\rightarrow$ lab detector measures frequency ν_0 identical to emitted frequency
- observer on ground
  $\rightarrow$ reaches detector at $t = h/c$

gravitational redshift

- detector has speed $v = gt = gh/c$
- $\rightarrow$ should have detected *blueshifted* frequency $> \nu_0$
- slow free-fall:

$$\frac{\Delta\nu}{\nu_0} = \frac{v}{c} = \frac{gh}{c^2}$$

gravitational redshift

- but detector found ν_0
  $\rightarrow$ curved space-time must have exactly compensated the shift by a *gravitational redshift* of

$$\frac{\Delta\nu}{\nu_0} = -\frac{v}{c} = -\frac{gh}{c^2}$$

- total gravitational redshift for light escaping to infinity
  $\rightarrow$ integrate with $g = GM/r^2$ and $dr = h$ (assume nearly flat space-time!)

$$\int_{\nu_0}^{\nu_\infty} \frac{d\nu}{\nu} \approx \int_{r_0}^{\infty} \frac{GM}{r^2 c^2}\, dr$$

gravitational redshift

- this gives:

$$\ln\left(\frac{\nu_\infty}{\nu_0}\right) \approx -\frac{GM}{r_0 c^2}$$

for *weak* gravity ($r_0/r_c = GM/r_0 c^2 \ll 1$).

gravitational redshift !!

- thus

$$\frac{\nu_\infty}{\nu_0} \approx \exp\left(-\frac{GM}{r_0 c^2}\right)$$

- with $\exp(-x) \approx 1 - x$ for $x \ll 1$ we get

$$\frac{\nu_\infty}{\nu_0} \approx 1 - \frac{GM}{r_0 c^2}$$

- *exact* result:

$$\frac{\nu_\infty}{\nu_0} = \left(1 - \frac{2GM}{r_0 c^2}\right)^{1/2}$$

gravitational redshift !!

- for the redshift z we get:

$$
\begin{aligned}
z &= \frac{\nu_0}{\nu_\infty} - 1 \\
&= \left(1 - \frac{2GM}{r_0 c^2}\right)^{-1/2} - 1 \\
&\approx \frac{GM}{r_0 c^2}
\end{aligned}
$$

gravitational time dilation

- clock with one tick per vibration of monochromatic light wave
- $\Delta t = 1/\nu$
- gravitational redshift
  $\rightarrow$ clock at r_0 will tick slower than clock at $r = \infty$

$$
\begin{aligned}
\frac{\Delta t_0}{\Delta t_\infty} &= \frac{\nu_\infty}{\nu_0} = \left(1 - \frac{2GM}{r_0 c^2}\right)^{1/2} \\
&\approx 1 - \frac{GM}{r_0 c^2}
\end{aligned}
$$

gravitational time dilation

- *time passes more slowly as the surrounding space-time becomes more curved*
- Example: Sirius B ($M = 2.1 \times 10^{33}\,\mathrm{g}$, $R = 5.5 \times 10^{8}\,\mathrm{cm}$)
  - $z \approx 2.8 \times 10^{-4}$
  - $\Delta t_{\infty} = 3600\,\mathrm{s} \rightarrow \Delta t_0 - \Delta t_{\infty} = 1\,\mathrm{s}$

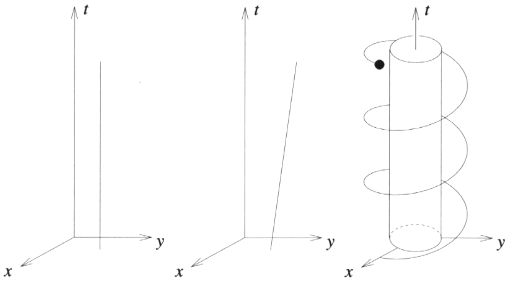
Intervals and Geodesics !!

- 4D space-time coordinates (x, y, z, ct) for each event
- field equations:

$$\mathcal{G} = -\frac{8\pi G}{c^4}\mathcal{T}$$

  - $\mathcal{T}$: stress-energy tensor
  - $\mathcal{G}$: Einstein tensor, describes curved space-time
- space-time diagrams:

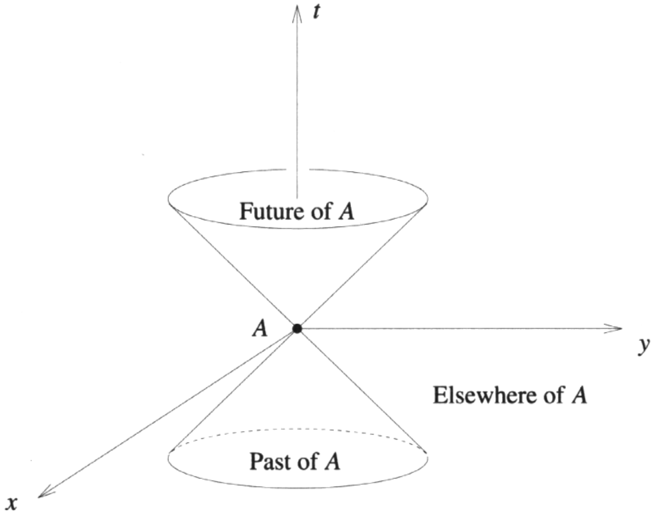
space-time diagrams



(a) (b) (c)

- (a) object at rest
- (b) object moving at constant speed
- (c) satellite orbiting planet
- *world-line*: path of object in space-time
- events and the *light cone*:

space-time diagrams



space-time diagrams

- distance in space-time: *space-time interval*

$$(\Delta s)^2 = [c(t_b - t_a)]^2 - (\mathbf{x_a} - \mathbf{x_b})^2$$

  in a flat space-time
- $(\Delta s)^2$ is *invariant* under Lorentz transformations
- $(\Delta s)^2$ can be positive, negative or zero

space-time diagrams

- $(\Delta s)^2 > 0$: *time-like interval*
  $\rightarrow$ light can travel between events a and b
  $\rightarrow$ can find inertial frame S that moves along a straight world-line connecting a and b so that both events happen at the same *location* in S (but at different times)
- *proper time*: interval between these events:

$$\Delta\tau = \frac{\Delta s}{c}$$

- $(\Delta s)^2 = 0$: *light-like* or *null* interval
  $\rightarrow$ proper time is zero!

space-time diagrams

- $(\Delta s)^2 < 0$: *space-like* interval
  light cannot travel between the events
  can find frames where events occur in different order or *simultaneously*
- *proper distance*: measure in frame where $t_a = t_b$:

$$\Delta \mathcal{L} = \sqrt{-(\Delta s)^2}$$

  would give the *rest length* of a rod connecting the events
- *metric* measures the differential distance along a path

$$(d\ell)^2 = (dx)^2 + (dy)^2 + (dz)^2$$

space-time diagrams

- integrating along the path (*line integral*)

$$\Delta \ell = \int_{p_1}^{p_2} \sqrt{(d\ell)^2} \quad \text{along } \mathcal{P}$$

- *metric for flat space-time*

$$(ds)^2 = (c\,dt)^2 - (dx)^2 - (dy)^2 - (dz)^2$$

- interval along a world-line $\mathcal{W}$:

$$\Delta s = \int_{p_1}^{p_2} \sqrt{(ds)^2} \quad \text{along } \mathcal{W}$$

space-time diagrams !!

- flat space-time
  $\rightarrow$ interval measured along straight time-like word-line between two events is *maximum*
- curved space-time: "straightest possible world lines"
  $\rightarrow$ *geodesics*

space-time diagrams !!

- time-like geodesics connecting 2 events
  $\rightarrow$ *extremum*
  $\rightarrow$ either *maximum* or *minimum* interval
- paths followed by free-falling particles are geodesics
- massless particles follow *null geodesics*
- *coordinate speed*: rate with which the spatial coordinates of an object change

space-time diagrams !!

- flat metric in spherical coordinates:

$$(d\ell)^2 = (dr)^2 + (r\, d\theta)^2 + (r \sin\theta\, d\phi)^2$$

- flat space-time metric in spherical coordinates:

$$(ds)^2 = (c\, dt)^2 - (dr)^2 - (r\, d\theta)^2 - (r \sin\theta\, d\phi)^2$$

Schwarzschild metric !!

- metric in the presence of a massive object:
  → *Schwarzschild metric*

$$
\begin{aligned}
(ds)^2 &= \left(c\,dt\sqrt{1-2GM/rc^2}\right)^2 - \left(\frac{dr}{\sqrt{1-2GM/rc^2}}\right)^2 \\
&\quad -(r\,d\theta)^2 - (r\sin\theta\,d\phi)^2
\end{aligned}
$$

- this is a *vacuum solution* of the field equations!
  → only valid *outside* the mass M!

Schwarzschild metric !!

- curvature $\rightarrow$ radial term
- radial distance between two simultaneous events ($dt = 0$):

$$d\mathcal{L} = \sqrt{-(ds)^2} = \frac{dr}{\sqrt{1 - 2GM/rc^2}}$$

- spatial distance $d\mathcal{L}$ is *larger* than coordinate distance dr!

Schwarzschild metric !!

- clock at rest at radial coordinate r
  $\rightarrow$ proper time $d\tau$

$$d\tau = \frac{ds}{c} = dt\sqrt{1 - 2GM/rc^2}$$

  $\rightarrow$ time passes slower than without the mass M

Schwarzschild metric

- *Example*: orbit of satellite around planet
- strict calculation delivers orbit and conservation laws in one sweep
- simplistic approach: satellite orbits around equator of Earth ($\theta = 90^\circ$) with specified angular speed $\omega = v/r$

Schwarzschild metric

- inserting $dr = 0$, $d\theta = 0$ and $d\phi = \omega\, dt$ into the Schwarzschild metric

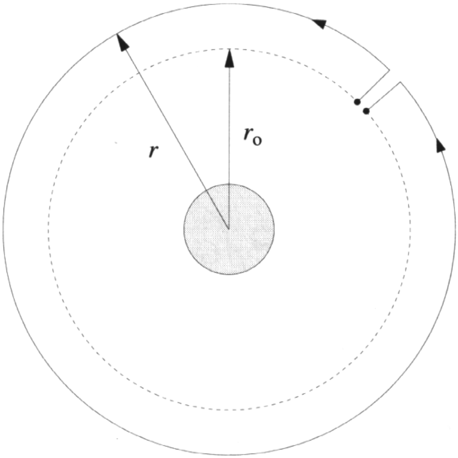
$$
\begin{aligned}
(ds)^2 &= \left[\left(c\sqrt{1 - 2GM/rc^2}\right)^2 - r^2\omega^2\right] dt^2 \\
&= \left(c^2 - \frac{2GM}{r} - r^2\omega^2\right) dt^2
\end{aligned}
$$

- integrating $\rightarrow$

$$
\Delta s = \int_0^{2\pi/\omega} \sqrt{c^2 - \frac{2GM}{r} - r^2\omega^2}\, dt
$$

Schwarzschild metric

- need to find extremum of this expression!
  $\rightarrow$ endpoints of word-line must be fixed:



Schwarzschild metric

- extremum $\rightarrow$

$$\frac{d}{dr}\Delta s = \frac{d}{dr}\left(\int_0^{2\pi/\omega} \sqrt{c^2 - \frac{2GM}{r} - r^2\omega^2}\, dt\right) = 0$$

Schwarzschild metric

- therefore:

$$\frac{d}{dr}\sqrt{c^2 - \frac{2GM}{r} - r^2\omega^2} = 0$$

so that

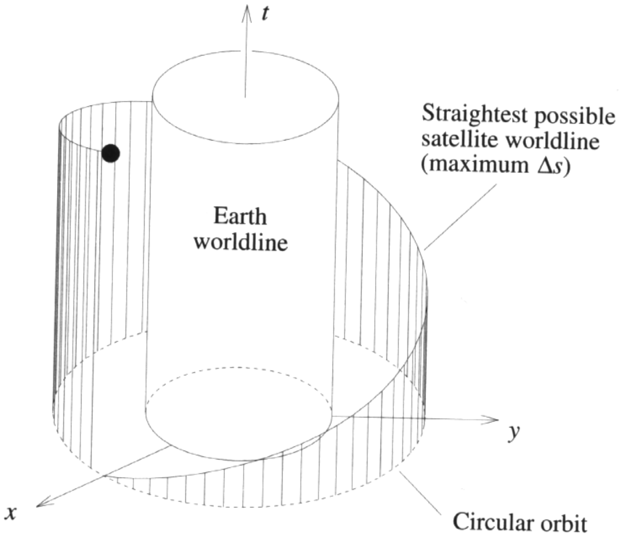
$$\frac{2GM}{r^2} - 2r\omega^2 = 0$$

or

$$v = r\omega = \sqrt{\frac{GM}{r}}$$

is the coordinate speed of the satellite

Schwarzschild metric



Black Holes !!

- old idea derived 1783 by amateur astronomer George Michell using Newton's particle model of light
- $\sqrt{}$ in Schwarzschild metric go to zero if

  $$R_S = 2GM/c^2$$

  is the surface radius of the object
  $\rightarrow$ *Schwarzschild radius*
- at R_S a clock would measure a proper time $d\tau = 0$

Black Holes

- apparent speed of light $\rightarrow$ coordinate speed of light $\rightarrow$ with $ds = 0$:

$$0 = \left(c\,dt\sqrt{1 - 2GM/rc^2}\right)^2 - \left(\frac{dr}{\sqrt{1 - 2GM/rc^2}}\right)^2 - (r\,d\theta)^2 - (r\sin\theta\,d\phi)^2$$

Black Holes

- $\rightarrow$ coordinate speed of a radially traveling photon ($d\theta = d\phi = 0$):

$$\frac{dr}{dt} = c\left(1 - \frac{2GM}{rc^2}\right) = c\left(1 - \frac{R_S}{r}\right)$$

Black Holes

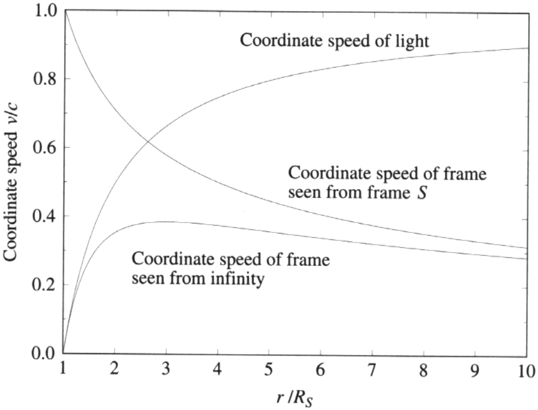


Figure 16.19 Coordinate speed of light, and coordinate speeds of a freely falling frame S seen by an observer at rest at infinity and by an observer in the frame S. The radial coordinates are in terms of R_S for a 10 $M_\odot$ black hole having a Schwarzschild radius of $\approx$ 30 km.

Black Holes !!

- at $r = R_S \rightarrow dr/dt = 0$
  $\rightarrow$ *Event horizon* of a *black hole*
- properties *inside* the BH cannot be observed but calculated
- center of a non-rotating BH $\rightarrow$ *singularity* with *all* of the mass of the BH

Black Holes

- free falling photon: integrate metric to obtain coordinate speed

$$\Delta t = \int_{r_1}^{r_2} \frac{dr}{dr/dt} = \frac{r_2 - r_1}{c} + \frac{R_S}{c} \ln\left(\frac{r_2 - R_S}{r_1 - R_S}\right)$$

  for $r_1 < r_2$
- $r_1 = R_S \rightarrow \Delta t = \infty$!

Black Holes

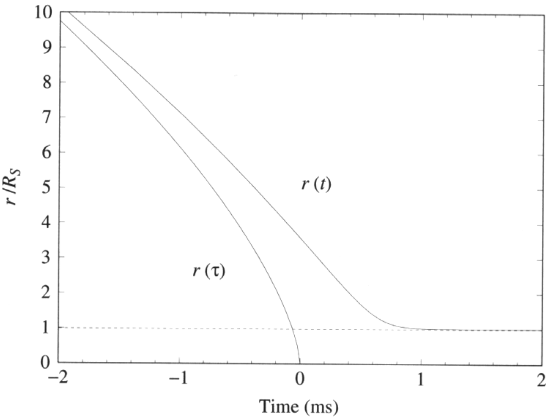


Figure 16.20 Coordinate $r(t)$ of a freely falling frame S according to an observer at rest at infinity, and $r(\tau)$ according to an observer in the frame S. The radial coordinates are in terms of R_S for a 10 $M_\odot$ black hole.

Black Holes !!

- $r(\tau)$: observer in free falling frame S
- $r(t)$: observer at rest at ∞
- object at rest at $r < R_S$: $dr = d\theta = d\phi = 0$

$$(ds)^2 = (c\,dt)^2 \left(1 - \frac{R_S}{r}\right) < 0$$

$\rightarrow$ space-like interval $\rightarrow$ not allowed for particles
$\rightarrow$ *impossible for particles to remain at rest for* $r < R_S$

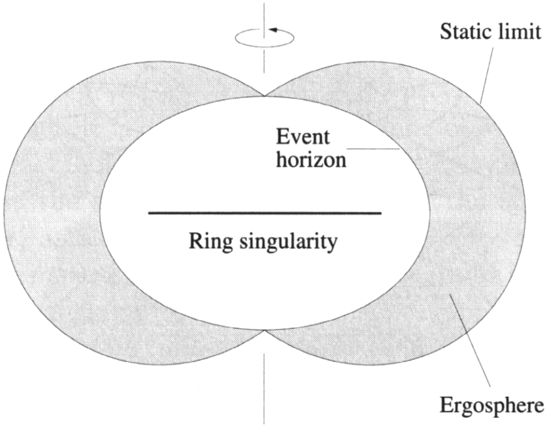
Black Holes

- non-rotating BH: all word-lines converge at the singularity
- after singularity formed exterior follows Schwarzschild metric
- maximum value of the angular momentum

$$L_{\mathrm{max}} = \frac{GM^2}{c}$$

Rotating Black Holes

- structure of a maximally rotating BH



Rotating Black Holes

- structure of metric changes $\rightarrow$ *frame dragging*
- *ring singularity*
- *ergosphere*: any particle *must* move in the direction of rotation of the BH
- frame dragging planned to be measured for Earth!